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General subtraction method for numerical calculation of one-loop QCD matrix elements

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Abstract: We present a subtraction scheme for eliminating the ultraviolet, soft, and collinear divergences in the numerical calculation of an arbitrary one-loop QCD amplitude with an arbitrary number of external legs. The subtractions consist of local counter terms in the space of the four-dimensional loop momentum. The ultraviolet subtraction terms reproduce $\overline{\rm MS}$ renormalization. The key point in the method for the soft and collinear subtractions is that, although the subtraction terms are defined graph-by-graph and the matrix element is also calculated graph-by-graph, the sum over graphs of the integral of each the subtraction term can be evaluated analytically and provides the well known simple pole structure that arises from subtractions from real emission graphs, but with the opposite sign.

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Contents

1. Introduction

Tests of the Standard Model or one of its extensions frequently involve high momentum transfer processes in which one is looking for effects from new interactions or particles. In most cases, at least some of the particles interact via the strong interactions. Then, in order to obtain reasonably accurate predictions for the expected cross sections, it is necessary to calculate the cross section at next-to-leading order (NLO) in quantum chromodynamics (QCD). Sometimes this is possible using the currently available theoretical tools, sometimes not. This paper concerns a method for potentially extending the range of problems for which a next-to-leading order calculation is possible.

An NLO calculation necessarily involves a virtual loop integration. That is, one has an integral

$$\int \frac{d^d l}{(2\pi)^d}\, \widetilde{\Gamma}(k_1, \ldots, k_n; l) \tag{1.1}$$

over a loop momentum l in $d = 4 - 2\epsilon$ dimensions, with momenta $k_1, \cdots, k_n$ leaving the graph at vertices around the loop. Of course, one wants $d = 4$ in the end, but there can be infrared divergences, which are temporarily regulated by working with $d \neq 4$. (There can also be ultraviolet divergences, but we ignore these for this initial discussion.) The infrared divergences arise because we work with a gauge theory containing massless particles. When some or all of the k_i are lightlike, the integral for $d = 4$ can have divergences arising from regions of l-space in which the momenta of two of the propagators in the loop approach a line parallel to one of the external momenta or in which the momentum of one of the propagators in the loop approaches zero. This leads to poles of the form $1/\epsilon^2$ and $1/\epsilon$ in the integral.

The traditional method of dealing with integrals like (1.1), initiated in the context of collider physics cross sections by Ellis, Ross, and Terrano [1], is to calculate the integrals analytically. The result is expressed in the form of the residues of the $1/\epsilon^2$ and $1/\epsilon$ poles and a remaining finite piece, which contains the most important physical information. It would seem not to be helpful to have a result with $1/\epsilon^2$ and $1/\epsilon$ terms, but it is helpful. There are other integrations involved in the calculation of a physical observable. These are integrations over the momenta of final state quarks and gluons in Feynman graphs without virtual loops. As long as the observable has the property known as infrared safety, the integration over final state momenta will produce $1/\epsilon^2$ and $1/\epsilon$ poles that precisely cancel the poles from the virtual loop integrals. The integrations over final state momenta are generally much too complicated to perform analytically. One can subtract from the integrand certain simple integrands that match the complete integrand in one of the soft or collinear limits that gives a divergence. Then the difference is integrable in $d = 4$ dimensions and this integral is performed numerically. The integral of each subtraction term is added back, but this time with the integration performed analytically. This gives $1/\epsilon^2$ and $1/\epsilon$ poles that cancel against the poles from the virtual loop integrals.

In order to make this method work, one needs analytical calculations of the virtual loop integrals that appear. Ellis, Ross, and Terrano [1] supplied the one-loop graphs with $n = 2$, $n = 3$, and $n = 4$ external legs that occur in a calculation for $e^+ + e^- \to 3\ jets$. Ellis and Sexton [2] supplied the additional $n = 2$, $n = 3$, and $n = 4$ graphs that are needed for $p + \bar{p} \to 2\ jets$. More recently, there have been heroic calculations by Bern, Dixon, and Kosower [3]–[6] and by Kunszt, Signer, and Trócsayni [7] for $n = 5$. With these results, one can calculate, for instance, $e^+ + e^- \to 4\ jets$ and $p + \bar{p} \to 3\ jets$. There are also calculations available with massive particles, such as massive quarks.

One wonders if we could not get beyond $n = 5$ by simply performing the virtual loop integrals numerically. Not only might one be able to get to diagrams with more external legs, but also one could have the flexibility to change the lagrangian, add masses, make approximations, and so forth by simply altering the integrand. Of course, one is not going to perform integrals in $d = 4 - 2\epsilon$ dimensions numerically. Instead, one would have to do something about the infrared divergences first, then take $d = 4$.

Something like this was tried in ref. [8]–[11] in the case of $e^+ + e^- \to 3\ jets$. There, the approach was to perform the integral over the energy l^0 in eq. (1.1) analytically and then to put all of the integrals over loop momenta $\vec{l}$ and final state particle momenta $\vec{p}$ together, summing over different contributions to the observable inside the integration. Then the real-virtual cancellations alluded to above happen inside the integrals, so that the integrals are convergent in three space dimensions. All of the integrals can be performed at once by numerical integration.

In this paper, we present a different proposal, one that is very close to the traditional method. One would subtract from the integrand (1.1) certain simple integrands that match the complete integrand in one of the soft or collinear limits that gives a divergence. Then the difference would be integrable in $d = 4$ dimensions and this integral would be performed numerically. The integral of each subtraction term would be added back, but this time with the integration performed analytically. This gives the same $1/\epsilon^2$ and $1/\epsilon$ poles that one has in an analytic calculation, ready for cancellation as usual.

We should mention that performing the 4-dimensional loop integral numerically is not completely without difficulties. In the massless theory, the integrand contains singularities $1/(p^2 + \mathrm{i}0)$ that lie on cones in the loop momentum space. One must deform the integration contour to avoid these singularities, as indicated by the "+i0" prescription. However, this same problem occurs in refs. [8]–[11], where it is handled in a straightforward manner.

The issue addressed in this paper is to find what one should subtract from the integrand. We treat this problem in QCD with massless quarks, leaving the problem of other lagrangians and non-zero masses for future work. We find that there is a set of subtractions that is quite simple in the sense that the subtraction terms can be easily generated by the same sort of computer algebra that generates the original integrand in (1.1). Furthermore, adding back the subtraction terms is straightforward: the residues of the $1/\epsilon^2$ and $1/\epsilon$ poles are the familiar ones, while the finite term that comes along with the poles is simple.

There has been other recent work aimed at automating the calculation of loop integrals in such a way that those integrals that are not amenable to analytic evaluation could be evaluated numerically [12, 13, 14]. This approach makes use of the Feynman parameter representation of the diagrams and is quite ambitious in that it applies to multiloop diagrams. In the present paper we deal only with one-loop diagrams, although with an arbitrary number of external legs. Our treatment is then to subtract from the integrands certain terms that remove divergences, but to otherwise leave the integrand entirely as given by the momentum space Feynman rules. Thus we aim at maximal simplicity of the method.

2. Strategy

We seek a method for performing subtractions on one-loop graphs in such a way that we are left with an integral that can be performed numerically in four dimensions. That means that we must take care of ultraviolet, collinear, and soft divergences.

We note first of all that in the case of tree graphs it is common to sum all of the graphs that contribute to a given set of external particles. Many graphs may contribute, but one can sum these graphs algebraically before entering into a numerical computation. With loop graphs, the situation is different. If we are going to integrate over the loop momentum numerically, we (or our computers) have to know the structure of the integrand, which is very different for different graphs. Thus we will work graph by graph.

JHEP09(2003)055

Working graph by graph, the choice of gauge matters. We take the simplest choice, the Feynman gauge. The external lines in each graph represent physical, on-shell particles. Thus our algorithms are based on having spinors $u(p,s)$, $\bar{u}(p,s)$, $v(p,s)$ and $\bar{v}(p,s)$ for external quarks and antiquarks and polarization vectors $\varepsilon^\mu(p,s)$ for external gluons. The polarization vectors can be defined using any physical gauge one cares to use. Our analysis makes use of the relation $p\cdot\varepsilon(p,s)=0$. If it were desired, one could use unphysical gluon polarizations, with $\sum_s \varepsilon^\mu(p,s)\varepsilon^\nu(p,s)^* \longrightarrow -g^{\mu\nu}$. Then the relation $p\cdot\varepsilon(p,s)=0$ fails, and the reader is asked to substitute the following argument. We calculate a one-loop amplitude, which must be multiplied by a tree level amplitude to calculate a term in a cross section. Where p_μ appeared contracted with a polarization vector, it now appears contracted with the tree level amplitude. But the contraction of p_μ with the tree level amplitude vanishes.

Thus our aim is to examine one-loop graphs with on-shell massless external particles. For each such graph, there may be ultraviolet, soft, and collinear divergences. In sections 4, 5, and 6, respectively, we will define subtractions for the integrand that eliminate each divergence. The ultraviolet subtractions are required for modified minimal subtraction ($\overline{\rm MS}$) renormalization. All that we have to do is to express the $\overline{\rm MS}$ counterterm as a subtraction on the integrand. In the case of the soft and collinear subtractions, we do not want to change the value of the graph, so we add back what we subtracted. However, we subtract something from the integrand and add back its integral summed over graphs. What we need to add back is *not* simple graph by graph. However, when we sum over graphs we get a simple prescription for what to add back. We begin our investigations in the next section by setting up some useful notation.

3. Matrix elements at tree and one-loop level

In this section, we set up our notation. First, we define a notation for the color and spin structure of matrix elements, following in general the notation of Catani and Seymour [15]. Then we add some notation specific to one-loop graphs. Finally, we define the wave function renormalization factors needed to relate amputated Green functions to the S-matrix.

The matrix element with m external QCD partons has the following general structure:

$$M^{c_1,\dots,c_m;s_1,\dots,s_m}(p_1,\dots,p_m)\,, \tag{3.1}$$

where $\{c_1,\dots,c_m\}$, $\{s_1,\dots,s_m\}$ and $\{p_1,\dots,p_m\}$ are respectively color indices (which take N_c^2-1 values for each gluon and N_c values for each quark or anti-quark), spin indices (which take $d-2$ values for gluons in d-dimensional spacetime, and 2 values for quarks and antiquarks) and momenta. We take the momenta p_i to be coming out of the graph, so that $p_i^2=0$ with $p_i^0>0$ for outgoing partons and $p_i^0<0$ for incoming partons. The matrix element is computed by contracting amputated Green functions with $\bar{u}(p,s)$ for an outgoing quark, $v(p,s)$ for an outgoing antiquark, and $\varepsilon(p,s)^*$ for an outgoing gluon. For incoming partons, the construction is analogous, using $u(-p,s), \bar{v}(-p,s)$ and $\varepsilon(-p,s)$. (There are identities available [16, 17] to simplify the spin wave functions, but these do not concern us in this paper.)

It is useful to represent the matrix element as a vector $|\mathcal{M}\rangle$ in color $\otimes$ spin space. One introduces a basis $\{|c_1,\ldots,c_m\rangle \otimes |s_1,\ldots,s_m\rangle\}$ in color $\otimes$ spin space in such a way that

$$M^{c_1,\ldots,c_m;s_1,\ldots,s_m}(p_1,\ldots,p_m) \equiv \Big(\langle c_1,\ldots,c_m| \otimes \langle s_1,\ldots,s_m|\Big)|\mathcal{M}(p_1,\ldots,p_m)\rangle . \tag{3.2}$$

With this notation, the matrix element squared (summed over final-state colors and spins) $|M|^2$ can be written as

$$|M|^2 = \langle \mathcal{M}(p_1,\ldots,p_m)|\, \mathcal{M}(p_1,\ldots,p_m)\rangle . \tag{3.3}$$

3.1 Color space

Following the notation of [15], we associate a color charge operator $\mathbf{T}_i$ with the emission of a gluon from each parton i. Its action onto the color space is defined by

$$\mathbf{T}_i = T^a_{cb}|a\rangle|c\rangle_{i\,i}\langle b| , \tag{3.4}$$

where T^a_{cb} is the appropriate color matrix (if_{cab} if the emitting particle i is a gluon, $t^a_{\alpha\beta}$ if the emitting particle i is a quark and $\bar{t}^a_{\alpha\beta} = -t^a_{\beta\alpha}$ if the emitting particle i is an antiquark). That is,

$$\langle c_1,\ldots,c_i,\ldots,c_m,c|\mathbf{T}_i|b_1,\ldots,b_i,\ldots,b_m\rangle = \delta_{c_1b_1}\cdots T^c_{c_ib_i}\cdots\delta_{c_mb_m} . \tag{3.5}$$

One also defines the corresponding adjoint operators $\mathbf{T}_i^\dagger$ for absorbing a gluon onto parton i. Then one defines a dot product of operators, $\mathbf{T}_i\cdot\mathbf{T}_j \equiv \mathbf{T}_i^\dagger\mathbf{T}_j$ for emitting a gluon from line i and absorbing it on line j. For $i \neq j$ this is

$$\mathbf{T}_i\cdot\mathbf{T}_j = \mathbf{T}_j\cdot\mathbf{T}_i = T^a_{c_ib_i}T^a_{c_jb_j}|c_i\rangle_i\,|c_j\rangle_j\;{}_i\langle b_i|_j\langle b_j| . \tag{3.6}$$

The case $i = j$ is special:

$$\mathbf{T}_i\cdot\mathbf{T}_i = C_i , \tag{3.7}$$

where C_i is the Casimir operator, that is, $C_i = C_A = N_c$ if i is a gluon and $C_i = C_F = (N_c^2-1)/2N_c$ if i is a quark or anti-quark.

Note that, by definition, each vector $|\mathcal{M}\rangle$ is a color singlet. Therefore color conservation is simply

$$\sum_{i=1}^m \mathbf{T}_i\,|\mathcal{M}(p_1,\ldots,p_m)\rangle = 0 . \tag{3.8}$$

Using this notation, we also define the square of color correlated amplitudes, $|\mathcal{M}^{i,k}|^2$, as follows

$$\begin{aligned}|\mathcal{M}^{i,k}|^2 &\equiv \langle\mathcal{M}(p_1,\ldots,p_m)|\,\mathbf{T}_i\cdot\mathbf{T}_k\,|\mathcal{M}(p_1,\ldots,p_m)\rangle \\ &= \Big[\mathcal{M}^{a_1\ldots b_i\ldots b_k\ldots a_m}(p_1,\ldots,p_m)\Big]^*\; T^c_{b_ia_i}\,T^c_{b_ka_k}\;\mathcal{M}^{a_1\ldots a_i\ldots a_k\ldots a_m}(p_1,\ldots,p_m) .\end{aligned} \tag{3.9}$$

3.2 One-loop graphs

As discussed in the introduction of this section, we wish to calculate the one-loop, m-parton matrix element graph by graph using numerical integration. In this subsection, we define some tools for this calculation. We use the vector notation in color and spin space of the previous subsection to write the one-loop matrix element as a sum of the one-loop Feynman graphs,

$$\big|\mathcal{M}(p_1,\dots,p_m)\big\rangle = \sum_{\text{graphs } G} \big|\mathcal{G}(G;p_1,\dots,p_m)\big\rangle . \tag{3.10}$$

The $|\mathcal{G}\rangle$ vector is related to the one-loop, amputated Green function contracted with the appropriate spinors and gluon polarization vectors by

$$\mathcal{G}^{c_1,\dots,c_m;s_1,\dots,s_m}(G;p_1,\dots,p_m) \equiv \Big(\big\langle c_1,\dots,c_m\big| \otimes \big\langle s_1,\dots,s_m\big|\Big)\big|\mathcal{G}(G;p_1,\dots,p_m)\big\rangle . \tag{3.11}$$

The graph G contains a one-loop, one-particle-irreducible subgraph Γ with $n = n(G)$ external lines and n internal lines around the loop. Let us label momenta of the internal lines $l_1, l_2, \dots l_n$. We integrate over the loop momentum. One can consider l_k for some specific index k to be the independent loop momentum, which we call l.[1] Once we make a choice of l_k as the independent variable, any of the propagator momenta around the loop is related to l by an equation of the form

$$l_i = l_k + \sum_{j=1}^{m} A(G,k)_{ij}\, p_j\,, \tag{3.12}$$

where the matrix A is determined by the structure of the graph and by our choice $l = l_k$. We write the integration over loop momentum as

$$\int \frac{d^d l}{(2\pi)^d}\,. \tag{3.13}$$

Since the integration measure in the space of loop momenta does not depend on the choice of origin (e.g. $dl_{k'} = dl_k$) we omit a label on dl indicating the definition of l. In the actual numerical integration, one chooses a definition for l, then chooses a random value for l thus defined, and then computes the l_i from eq. (3.12). Thus our notation is

$$\big|\mathcal{G}(G;p_1,\dots,p_m)\big\rangle = \int \frac{d^d l}{(2\pi)^d}\,\big|\widetilde{\mathcal{G}}(G;\{l\};p_1,\dots,p_m)\big\rangle\,, \tag{3.14}$$

where $\widetilde{\mathcal{G}}$ denotes the integrand for graph G and $\{l\}$ is a shorthand for $l_1,\dots,l_n$.

The loop integral could be ultraviolet divergent. This happens for $n = 2,3$ and for $n = 4$ in the case that all of the lines entering the loop are gluons. In this case we define a suitable UV counterterm that reproduces the same result as $\overline{\text{MS}}$ renormalization. We define the counter terms in section 4.

[1] Alternatively, l could be the average of the l_k. This choice is convenient for constructing the ultraviolet counterterm for a loop that needs ultraviolet renormalization. See section 4.

In certain graphs, the loop integration is infrared divergent. For instance when an internal gluon line connects two external lines the integral has soft, collinear, and soft-collinear poles. To handle the infrared divergences, we introduce simple counter terms that we subtract from the integrand for the numerical integration and then add back analytically. We discuss this in section 5 for the soft divergences and in section 6 for the collinear divergences.

3.3 S-matrix

The S-matrix can be constructed from the amputated Green functions by means of the LSZ reduction formula

$$\big|\mathcal{M}_{\rm full}(p_1,\dots,p_m)\big\rangle = \left(\prod_i \sqrt{r_i}\right) \big|\mathcal{M}^{(T)}_{\rm full}(p_1,\dots,p_m)\big\rangle\,. \tag{3.15}$$

Here we consider both sides of the equation to be vectors in color and spin space. The subscripts "full" indicate that this formula applies to the matrix element summed over orders of perturbation theory. The T (for "truncated") superscript on $\mathcal{M}$ on the right indicates that the matrix element is calculated by multiplying amputated Green functions by the appropriate Dirac spinors and polarization vectors. The factors $\sqrt{r_i}$, one for each external particle, are of two types, one for gluons and one for quarks and antiquarks. They are defined from the residues of the poles of the corresponding propagators at $p^2 = 0$. Specifically, when the flavor of particle i is q or $\bar{q}$ then $r_i = R_\psi$, where the renormalized quark propagator near $p^2 = 0$ has the behavior

$$iS(p) \sim R_\psi \frac{i\not{p}}{p^2 + {\rm i}0} \qquad p^2 \to 0\,. \tag{3.16}$$

When the flavor of particle i is g then $r_i = R_A$ where the renormalized gluon propagator near $p^2 = 0$ has the behavior

$$iD^{\mu\nu}(p) \sim R_A \frac{-ig^{\mu\nu}}{p^2 + {\rm i}0} + \cdots \qquad p^2 \to 0\,, \tag{3.17}$$

where the omitted terms are gauge terms proportional to p^μ or p^ν.

The constants r_i have perturbative expansions

$$r_i = 1 + r_i^{(1)} + \cdots\,, \tag{3.18}$$

where $r_i^{(1)} \propto \alpha_s^1$. Thus, including terms up to one-loop order,

$$\begin{aligned}\big|\mathcal{M}_{\rm full}(p_1,\dots,p_m)\big\rangle = {} & \big|\mathcal{M}_{\rm tree}(p_1,\dots,p_m)\big\rangle + \big|\mathcal{M}^{(T,1)}(p_1,\dots,p_m)\big\rangle + \\ & + \sum_i \frac{1}{2} r_i^{(1)} \big|\mathcal{M}_{\rm tree}(p_1,\dots,p_m)\big\rangle\,.\end{aligned} \tag{3.19}$$

Here $\big|\mathcal{M}^{(T,1)}\big\rangle$ is the one-loop contribution to $\big|\mathcal{M}^{(T)}\big\rangle$. Elsewhere in this paper we denote $\big|\mathcal{M}^{(T,1)}\big\rangle$ simply as $\big|\mathcal{M}\big\rangle$.

Now we need to evaluate the residue constants $r_i^{(1)}$. Consider first the quark propagator. Expanding to order α_s we have

$$iS(p) = \frac{\mathrm{i}\not{p}}{p^2 + \mathrm{i}0} + \frac{\mathrm{i}\not{p}}{p^2 + \mathrm{i}0}\,(-\mathrm{i}\Sigma_R(p))\frac{\mathrm{i}\not{p}}{p^2 + \mathrm{i}0}\,, \tag{3.20}$$

where $\Sigma_R(p)$ is the one-loop renormalized quark self-energy graph. It has the form

$$\Sigma_R(p) = A(p^2)\,\not{p} - Z_\psi^{(1)}\,\not{p}\,, \tag{3.21}$$

where evaluation of the graph in $d - 4 - 2\epsilon$ dimensions gives

$$A(p^2) = -C_F\,\frac{\alpha_s}{4\pi}\,\frac{1}{\epsilon}\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,\left(\frac{\mu^2}{-p^2}\right)^\epsilon\,\frac{\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)^3\,(1-\epsilon)}{\Gamma(1-2\epsilon)\,(1-2\epsilon)} \tag{3.22}$$

and where $Z_\psi^{(1)}$ is the one-loop $\overline{\mathrm{MS}}$ counter term,

$$Z_\psi^{(1)} = -C_F\,\frac{\alpha_s}{4\pi}\,\frac{1}{\epsilon}\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,. \tag{3.23}$$

Inserting this into eq. (3.20), we find

$$iS(p) = \frac{\mathrm{i}\not{p}}{p^2 + \mathrm{i}0}\left\{1 + A(p^2) - Z_\psi^{(1)}\right\}. \tag{3.24}$$

We now want to take $p^2 \to 0$ in order to find R_ψ. The renormalization counter term $Z_\psi^{(1)}$ removes the pole at $\epsilon = 0$ for any negative value of p^2. Having removed the pole, we can analytically continue ϵ to $\mathrm{Re}\,\epsilon < 0$. Then we have $A(p^2) \to 0$ as $-p^2 \to 0$. This leaves

$$iS(p) \sim \frac{\mathrm{i}\not{p}}{p^2 + \mathrm{i}0}\left\{1 - Z_\psi^{(1)}\right\}. \tag{3.25}$$

Comparing to eq. (3.16), we have

$$R_\psi^{(1)} = -Z_\psi^{(1)} = C_F\,\frac{\alpha_s}{4\pi}\,\frac{1}{\epsilon}\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,. \tag{3.26}$$

The analysis for gluons is similar. One has

$$R_A^{(1)} = -Z_A^{(1)} = -\left[\frac{5}{3}\,C_A - \frac{4}{3}\,T_R\,n_f\right]\,\frac{\alpha_s}{4\pi}\,\frac{1}{\epsilon}\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\,. \tag{3.27}$$

where $T_R = 1/2$ and n_f is the number of quark flavors.

4. Renormalization

In the following sections, we will consider the divergences of infrared origin that can arise in the integral

$$\int \frac{d^d l}{(2\pi)^d}\,\big|\widetilde{\mathcal{G}}(G;\{l\};p_1,\ldots,p_m)\big\rangle \tag{4.1}$$

for a graph G. First, however, we should deal with the possible ultraviolet divergences. These are to be eliminated according to the $\overline{\rm MS}$ renormalization prescription. However, since we calculate loop integrals by numerical integration, the implementation of the $\overline{\rm MS}$ prescription needs some analysis. This is the subject of the present section.

Consider an ultraviolet divergent one-loop graph with n propagators in the loop. Here n could be 2, 3, or 4. Denote the momenta leaving Γ by $k_1, \dots, k_n$. In our application, Γ will be a subgraph of G and the momenta $k_1, \dots, k_n$ will be linear combinations of $p_1, \dots, p_m$. The graph Γ has the generic form

$$\Gamma(k_1, \dots, k_n) = \int \frac{d^d l}{(2\pi)^d} \, \widetilde{\Gamma}(k_1, \dots, k_n; l) \,. \tag{4.2}$$

The functions Γ and $\widetilde{\Gamma}$ may carry spinor and vector indices as well as color indices. Our notation here suppresses these indices. We seek to calculate the renormalized version of Γ. With $\overline{\rm MS}$ renormalization, this is

$$[\Gamma]_{\rm R} = \lim_{\epsilon \to 0} \left\{ \Gamma - [\Gamma]_{\rm pole} \right\} , \tag{4.3}$$

where

$$[\Gamma]_{\rm pole} = \frac{1}{\epsilon} \, \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \; \times \lim_{\epsilon \to 0} \left[\epsilon \, \Gamma(\epsilon) \right] . \tag{4.4}$$

Here we are to choose the external momenta so that Γ does not have infrared divergences. For instance, all of the external momenta can be spacelike. Then the only pole present in $\Gamma(\epsilon)$ in eq. (4.4) is the ultraviolet pole that is to be removed by renormalization.

Since we are performing integrals numerically, we need to represent $[\Gamma]_{\rm pole}$ as an integral

$$[\Gamma]_{\rm pole} = \int \frac{d^d l}{(2\pi)^d} \, \widetilde{\Gamma}_{UV}(k_1, \dots, k_n; l) \tag{4.5}$$

in such a way that $[\Gamma]_R$ can be calculated as

$$[\Gamma]_R = \int \frac{d^4 l}{(2\pi)^4} \, \lim_{\epsilon \to 0} \left\{ \widetilde{\Gamma}(k_1, \dots, k_n; l) - \widetilde{\Gamma}_{UV}(k_1, \dots, k_n; l) \right\} . \tag{4.6}$$

This last step is justified if the integrands $\widetilde{\Gamma}$ and $\widetilde{\Gamma}_{UV}$ match up to an l^{-5} remainder for $l \to \infty$ at fixed ϵ and if $\widetilde{\Gamma}_{UV}$ if free of infrared singularities.

There is more than one possibility for $\widetilde{\Gamma}_{UV}$. However, there is a simple prescription that works in all cases save one, the one-loop gluon self-energy, which has a quadratic divergence and needs a more elaborate treatment. The prescription, ignoring complications arising from the tensor structure of Γ, is as follows. (We give the full details including the gluon self-energy and the tensor structure in appendix A). Write the starting integrand in the form

$$\widetilde{\Gamma}(k_1, \dots, k_n; l) = \frac{N(k_1, \dots, k_n; l)}{(l_1^2 + {\rm i}0) \cdots (l_n^2 + {\rm i}0)} \,. \tag{4.7}$$

Here we display the denominator for each propagator and put everything else in a numerator function $N(k_1, \dots, k_n; l)$. We make a definite choice for the loop momentum l. In general,

$$l_i = l + K_i \,, \tag{4.8}$$

where K_i is a linear combination of the momenta k_j leaving the graph at the vertices. We define

$$l = \frac{1}{n}[l_1 + l_2 + \cdots + l_n]\,. \tag{4.9}$$

Then

$$\sum_i K_i = 0\,. \tag{4.10}$$

The numerator contains an overall factor $\mu^{2\epsilon}$ times a polynomial in l, with coefficients that can be polynomials in ϵ. Except for the gluon two-point function, which we treat separately, the highest order term is of order l^{2n-3}, corresponding to a linearly divergent integral for $d = 4$. In some cases, this term is absent. The next highest order term is of order l^{2n-4}, corresponding to a logarithmically divergent integral. This term is present as long as we are dealing with a graph that needs renormalization. Call these two terms N_{UV},

$$N(k_1, \ldots, k_n; l) = N_{UV}(k_1, \ldots, k_n; l) + \mathcal{O}(l^{2n-5})\,. \tag{4.11}$$

Now define

$$\widetilde{\Gamma}_{UV}(k_1, \ldots, k_n; l) = \frac{N_{UV}(k_1, \ldots, k_n; l)}{(l^2 - \mu^2 e^{\lambda} + \mathrm{i}0)^n}\,, \tag{4.12}$$

where μ is the $\overline{\mathrm{MS}}$ renormalization scale and λ is a constant to be determined. With this choice, $\widetilde{\Gamma}_{UV}$ is free of infrared singularities that could lead to an infrared divergence. It matches $\widetilde{\Gamma}$ for large l. Here it is evident that the leading terms, which correspond to the linear divergence, match. To treat the next terms, note that for $l \to \infty$,

$$\frac{1}{(l+K_1)^2 \cdots (l+K_n)^2} \sim \frac{1}{(l^2)^n} - \frac{\sum_i 2l \cdot K_i}{(l^2)^{n+1}} + \mathcal{O}\left(\frac{1}{(l^2)^{n+1}}\right). \tag{4.13}$$

The second term vanishes because of eq. (4.10). For this reason, the order l^{2n-4} term in $\widetilde{\Gamma} - \widetilde{\Gamma}_{UV}$, which would give a logarithmic divergence, vanishes. Thus $\widetilde{\Gamma} - \widetilde{\Gamma}_{UV}$ falls off fast enough that its integral is ultraviolet finite when d is near 4. Consequently, the $\epsilon \to 0$ limit of the integral of $\widetilde{\Gamma} - \widetilde{\Gamma}_{UV}$ can be evaluated by simply setting $d = 4$.

We note next that the integral

$$\int \frac{d^d l}{(2\pi)^d}\, \widetilde{\Gamma}_{UV}(k_1, \ldots, k_n; l) \tag{4.14}$$

will be a constant times the $\overline{\mathrm{MS}}$ pole factor $(1/\epsilon)\,(4\pi)^\epsilon/\Gamma(1-\epsilon)$ plus a remaining finite piece. The constant multiplying the pole is necessarily the same constant as for the integral of $\widetilde{\Gamma}$ since the ultraviolet behaviors of Γ and Γ_{UV} match. The finite constant depends on λ, so it is a simple matter to choose λ such that the finite part vanishes, yielding eq. (4.5).

We have seen here how to construct the renormalization counterterm for a divergent subgraph Γ of a graph G. Using this counterterm, renormalization amounts to replacing

$$\int \frac{d^d l}{(2\pi)^d}\, \big|\widetilde{\mathcal{G}}(G; \{l\}; p_1, \ldots, p_m)\big\rangle \tag{4.15}$$

by

$$\int \frac{d^d l}{(2\pi)^d} \left\{ \left| \widetilde{\mathcal{G}}(G;\{l\};p_1,\ldots,p_m) \right\rangle - \left| \widetilde{\mathcal{R}}(G;\{l\};p_1,\ldots,p_m) \right\rangle \right\}, \tag{4.16}$$

where $\left|\widetilde{\mathcal{R}}(G;\{l\};p_1,\ldots,p_m)\right\rangle$ is obtained from G by substituting $\widetilde{\Gamma}_{UV}$ for $\widetilde{\Gamma}$. In the case that graph G is ultraviolet finite, we define $\left|\widetilde{\mathcal{R}}(G;\{l\};p_1,\ldots,p_m)\right\rangle$ to be zero. We will use this notation in subsequent sections.

We give specific expressions for $\widetilde{\Gamma}_{UV}(k_1,\ldots,k_n;l)$ in appendix A. There, we also exhibit what to do with the tensor structure of the $\widetilde{\Gamma}(k_1,\ldots,k_n;l)$ and with the gluon self-energy diagram, which needs a somewhat different treatment than presented above because it contains a quadratic divergence.

5. Soft singularities

The integrand of a one-loop graph becomes singular when the momentum of an internal gluon loop line that connects to two external lines becomes soft. Consider a one-loop graph G with external momenta $p_1, p_2, \ldots p_m$ directed out of the graph. Choose two of the external lines, with labels i and j. If the lines i and j connect via three-point vertices to the two ends of a gluon propagator in the loop, then we will say that $\{i,j\}$ is in the soft indices class for graph G, $\{i,j\} \in I_S(G)$. In this case the integrand $\left|\widetilde{\mathcal{G}}(G;\{l\};p_1,\ldots,p_m)\right\rangle$ is singular in the limit that the momentum of the gluon propagator that connects lines i and j, call it l_k, tends to zero. The loop integration is logarithmically divergent at this point.

The denominators that become singular in the soft limit $l_k \to 0$ are l_k^2 for the soft gluon propagator, $(l_k + p_i)^2$ for the immediately preceding propagator in the loop and $(l_k - p_j)^2$ for the immediately following propagator[2]. It is useful to define a soft limit function

$$\left| f_{ij}^S(G;p_1,\ldots,p_m) \right\rangle = \lim_{l_k \to 0} l_k^2 (l_k + p_i)^2 (l_k - p_j)^2 \left| \widetilde{\mathcal{G}}(G;l_1,\ldots,l_n;p_1,\ldots,p_m) \right\rangle. \tag{5.1}$$

Here we implicitly take l_k to be the independent loop momentum and use eq. (3.12) to determine the other l_i. With this definition, $\left|f_{ij}^S\right\rangle$ vanishes if $\{i,j\} \notin I_S(G)$. We can then define a soft subtraction for the integrand $\widetilde{\mathcal{G}}$ as

$$\left| \widetilde{\mathcal{S}}_{ij}(G;l_k;p_1,\ldots,p_m) \right\rangle = \frac{\left| f_{ij}^S(G;p_1,\ldots,p_m) \right\rangle}{(l_k^2 + \mathrm{i}0)((l_k+p_i)^2 + \mathrm{i}0)((l_k - p_j)^2 + \mathrm{i}0)}. \tag{5.2}$$

The integral for the original graph minus this subtraction is

$$\int \frac{d^d l}{(2\pi)^d} \left\{ \left| \widetilde{\mathcal{G}}(G;\{l\};p_1,\ldots,p_m) \right\rangle - \left| \widetilde{\mathcal{S}}_{ij}(G;l_k;p_1,\ldots,p_m) \right\rangle \right\}. \tag{5.3}$$

By construction, this integral does *not* have an infrared divergence from the region $l_k \to 0$. It may have other divergences, which will be eliminated by other subtraction terms. For instance, there may be another pair of external lines $\{i',j'\}$ that connect to the virtual loop

[2]Here we choose the positive direction around the loop to be from line i to line j.

and are joined by a virtual gluon line with label k'. Then we should also subtract $|\mathcal{S}_{i'j'}\rangle$, as defined above, from the graph. There may also be collinear divergences, to be discussed in the following section. Finally, there may be ultraviolet divergences, as discussed in section 4. The idea is that after all divergences are subtracted, the indicated loop integral can be performed by numerical integration.

Having subtracted the integral of $|\widetilde{\mathcal{S}}_{ij}\rangle$, we should add it back. For this purpose, we need to study the structure of the soft subtraction in more detail. The first step is to extend the vector notation that we have been using to cover unphysical gluon polarizations. Previously we have implicitly supplied a projection operator $|s\rangle\langle s|$ onto the $d-2$ dimensional space of physical polarizations for each external gluon. For the purpose of the present section we consider a d-dimensional polarization space.

In addition to the physical polarizations represented by $|s\rangle$ we introduce polarizations $|\mu\rangle$ along the coordinate axes. These have the orthogonality relations

$$\langle\mu|\nu\rangle = -g_{\mu\nu}\,. \tag{5.4}$$

The minus sign is present because we have chosen the inner product in the bra-ket spin space so that $\langle s'|s\rangle = +\delta_{s's}$. The corresponding completeness relation for the $|\mu\rangle$ vectors is

$$1 = -|\mu\rangle g^{\mu\nu}\langle\nu| \tag{5.5}$$

with an implicit summation. We take the sign convention for the vectors $|\mu\rangle$ such that

$$\langle\mu|\mathcal{M}\rangle = M_\mu\,. \tag{5.6}$$

Then our previous definition $\langle s|\mathcal{M}\rangle = \varepsilon^*(p,s)^\mu M_\mu$ can be written

$$\langle s|\mathcal{M}\rangle = \varepsilon^*(p,s)^\mu\langle\mu|\mathcal{M}\rangle\,. \tag{5.7}$$

The soft limit function has a rather simple form

$$\big|f_{ij}^S(G;p_1,\dots,p_m)\big\rangle = -\mathrm{i}g_s^2\mu^{2\epsilon}4p_i\!\cdot\!p_j\,\mathbf{T}_i\cdot\mathbf{T}_j\,\boldsymbol{H}_{ij}\big|\mathcal{G}(G_{ij}(G);p_1,\dots,p_m)\big\rangle\,, \tag{5.8}$$

where the graph $G_{ij}(G)$ represents the tree level amputated graph obtained from graph G by omitting the three singular propagators and the vertices where they join the external lines i and j. Here $\boldsymbol{H}_{ij}$ is an operator in the spin space of the partons i and j. The form of $\boldsymbol{H}_{ij}$ depends on whether partons i and j are quarks (or equivalently antiquarks) or gluons. In computing $\boldsymbol{H}_{ij}$, we keep only terms that survive under projecting onto physical polarizations for the final state partons. We find

$$\begin{aligned}
\boldsymbol{H}_{ij}(qq) &= 1\,,\\
\boldsymbol{H}_{ij}(qg) &= 1 + |\nu\rangle_j\,\frac{p_i^\nu p_j^\beta}{2p_i\cdot p_j}\,{}_j\langle\beta|\,,\\
\boldsymbol{H}_{ij}(gg) &= 1 + |\mu\rangle_i\,\frac{p_j^\mu\;p_i^\alpha}{2p_i\cdot p_j}\,{}_i\langle\alpha| + |\nu\rangle_j\,\frac{p_i^\nu p_j^\beta}{2p_i\cdot p_j}\,{}_j\langle\beta| + |\mu\rangle_i\,|\nu\rangle_j\,\frac{g^{\mu\nu}p_i^\alpha\,p_j^\beta}{4p_i\cdot p_j}\,{}_i\langle\alpha|\,{}_j\langle\beta|\,.
\end{aligned} \tag{5.9}$$

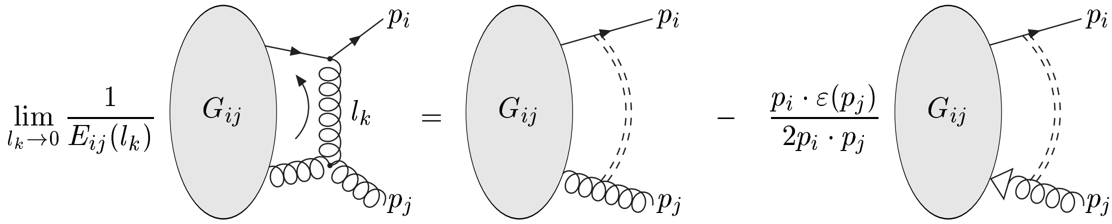

Figure 1: Pictorial representation of the soft gluon subtraction, eq. (5.10). The quantity depicted is $(1/E_{ij})\big|\mathcal{S}_{ij}\big\rangle$ for $i = q$, $j = g$. The double dashed line represents the octet color exchange contained in the factor $\mathbf{T}_i \cdot \mathbf{T}_j$. The two terms correspond to the two terms in $\boldsymbol{H}_{ij}(qg)$. In the second term, the arrow on the gluon line represents the factor $p^{\beta}_{j\,j}\big\langle\beta\big|$, which has the effect of replacing the polarization vector $\epsilon^*(p_j, s)$ that would normally be contracted with the amputated Green function by the momentum p_j. The polarization vector is then contracted with p_i. The minus sign arises from our sign conventions, in which $\big\langle s\big|\nu\big\rangle_j\, p_i^{\nu} = -\epsilon^*(p_j, s)\cdot p_i$.

Notice that in each term beyond the unit operator there is a factor $p^{\beta}_{j\,j}\big\langle\beta\big|$ or $p^{\alpha}_{i\,i}\big\langle\alpha\big|$. This feature will play an important role in the subsequent analysis.

With this result, the soft subtraction is

$$\big|\widetilde{\mathcal{S}}_{ij}(G; l_k; p_1, \ldots, p_m)\big\rangle = E_{ij}(l_k, p_i, p_j)\, \mathbf{T}_i \cdot \mathbf{T}_j\, \boldsymbol{H}_{ij}\, \big|\mathcal{G}(G_{ij}(G); p_1, \ldots, p_m)\big\rangle\,, \tag{5.10}$$

where the $E_{ij}(l_k, p_i, p_j)$ is the familiar eikonal factor [18, 19],

$$E_{ij}(l_k, p_i, p_j) = -\mathrm{i}g_s^2 \mu^{2\epsilon} \frac{4p_i \cdot p_j}{(l_k^2 + \mathrm{i}0)((l_k + p_i)^2 + \mathrm{i}0)((l_k - p_j)^2 + \mathrm{i}0)}\,. \tag{5.11}$$

Note that, in E_{ij}, one could arrange the definitions so that the denominator $(l_k+p_i)^2+\mathrm{i}0 = 2l_k \cdot p_i + l_k^2 + \mathrm{i}0$ is replaced by $2l_k \cdot p_i + \mathrm{i}0$, as in [18], since these expressions match in the limit $l_k \to 0$. Similarly $(l_k - p_j)^2 + \mathrm{i}0$ could have been $-2l_k \cdot p_j + \mathrm{i}0$. The form used in eq. (5.11) suits our purpose here because its integral over l_k is ultraviolet convergent. The subtraction term in eq. (5.10) is illustrated in figure 1.

We need to add back the sum over graphs of the integral $\big|\mathcal{S}_{ij}\big\rangle$ of $\big|\widetilde{\mathcal{S}}_{ij}\big\rangle$. We are now prepared to calculate what this quantity is. Defining $\mathcal{V}^{\rm soft}_{ij}(\epsilon)$ to be the integral of $E_{ij}(l)$, we have

$$\begin{aligned}\big|\mathcal{S}_{ij}(G; p_1, \ldots, p_m)\big\rangle &\equiv \int \frac{d^d l}{(2\pi)^d}\, \big|\widetilde{\mathcal{S}}_{ij}(G; \{l\}, p_1, \ldots, p_m)\big\rangle \\ &= \mathcal{V}^{\rm soft}_{ij}(\epsilon)\, \mathbf{T}_i \cdot \mathbf{T}_j\, \boldsymbol{H}_{ij}\, \big|\widetilde{\mathcal{G}}(G_{ij}(G); p_1, \ldots, p_m)\big\rangle\,. \end{aligned} \tag{5.12}$$

The integral can be performed analytically, with the result

$$\mathcal{V}^{\rm soft}_{ij}(\epsilon) \equiv \int \frac{d^d l}{(2\pi)^d}\, E_{ij}(l) = \frac{\alpha_s}{4\pi} \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{\mu^2}{-2p_i \cdot p_j}\right)^{\epsilon} \left(\frac{2}{\epsilon^2} + \mathcal{O}(\epsilon)\right). \tag{5.13}$$

Thus $\big|\mathcal{S}_{ij}(G)\big\rangle$ is a known integral times a tree level graph with a modified color factor and a kinematic factor $\boldsymbol{H}_{ij}$.

The sum over graphs G of the subtraction terms is even simpler. We consider a given set of final state parton flavors and a given choice of $\{i,j\}$. The tree graphs $G_{ij}(G)$ have the same parton flavors and momenta. Summing over all graphs G for which $\{i,j\} \in I_S(G)$, the graphs $G_{ij}(G)$ cover all tree graphs with this choice of external partons. Thus summing over graphs G gives the full tree amplitude $\big|\mathcal{M}_{\rm tree}(p_1,\dots,p_m)\big\rangle$ corresponding to the given final state. On the other hand the tree level matrix element is gauge invariant, that is

$$p^{\mu}_{j\,j}\big\langle\mu\big|\mathcal{M}_{\rm tree}(p_1,\dots,p_m)\big\rangle = 0\,, \tag{5.14}$$

where j is a gluon leg. Thus gauge invariance ensures that the complicated terms in the operator $\boldsymbol{H}_{ij}$, eq. (5.9), do not contribute, so that $\boldsymbol{H}_{ij}$ can be replaced by the unit operator. Then the soft contribution that we need to add back is

$$\sum_G \big|\mathcal{S}_{ij}(G;p_1,\dots,p_m)\big\rangle = \mathcal{V}^{\rm soft}_{ij}(\epsilon)\,\mathbf{T}_i\cdot\mathbf{T}_j\,\big|\mathcal{M}_{\rm tree}(p_1,\dots,p_m)\big\rangle\,. \tag{5.15}$$

We now summarize the soft gluon subtractions, this time including a sum over pairs of indices $\{i,j\}$. We subtract

$$\sum_{\{i,j\}\in I_S(G)} \big|\widetilde{\mathcal{S}}_{ij}(G;l_k;p_1,\dots,p_m)\big\rangle\,, \tag{5.16}$$

defined in eq. (5.2) from the integrand $\big|\widetilde{\mathcal{G}}(G;\{l\},p_1,\dots,p_m)\big\rangle$ for each one-loop graph G. Then we add the integrals of these terms back in the form

$$\sum_{\{i,j\}} \mathcal{V}^{\rm soft}_{ij}(\epsilon)\,\mathbf{T}_i\cdot\mathbf{T}_j\,\big|\mathcal{M}_{\rm tree}(p_1,\dots,p_m)\big\rangle\,. \tag{5.17}$$

In section 7, we will see that the terms added back are cancelled by similar terms corresponding to real soft gluon emission. The terms subtracted from each one-loop graph serve to remove the soft gluon divergences. This leaves the collinear parton divergences, which are the subject of the next section.

6. Collinear singularities

One-loop graphs have logarithmic infrared divergences that arise from integration regions in which the momentum on an internal loop line that connects to an external line becomes collinear with the momentum of the external line. In this section, we define a term that, when subtracted from the graph, eliminates the divergence. This construction is quite similar to what we needed for soft gluon divergences. However, the structure of the collinear subtraction term is more complicated, so we will need a more involved analysis.

Consider a one-loop graph G with external momenta $p_1, p_2, \dots p_m$ directed out of the graph. Choose one of the external lines, with label i. If the line i connects via a three-point vertex to the loop, and if the loop partons are both gluons or are one gluon and one quark or antiquark, then we will say that i is in the collinear index class for graph G, $i \in I_C(G)$. In this case the loop integration has a logarithmic divergence arising from the region in which the momenta of the loop propagators that connect to line i, call them j and $j+1$, become collinear to the outgoing momentum of line i,

$$\begin{aligned} l_j &\to x\, p_i \\ -l_{j+1} &\to (1-x)\, p_i\,, \end{aligned} \tag{6.1}$$

with $0 < x < 1$.

The denominators that become singular in the collinear limit (6.1) are l_j^2 and $(-l_{j+1})^2 = (p_i - l_j)^2$. The coefficient of $1/[l_j^2 \times (p_i - l_j)^2]$ in the integrand for the graph is non-singular in the collinear limit. It is useful to define a collinear coefficient function $|f_i^{C,0}(G;x;p_1,\ldots,p_m)\rangle$ by

$$\big|f_i^{C,0}(G;x;p_1,\ldots,p_m)\big\rangle = \lim_{l_j \to x\, p_i} l_j^2\, (p_i - l_j)^2\, \big|\widetilde{\mathcal{G}}(G;l_1,\ldots,l_n;p_1,\ldots,p_m)\big\rangle\,. \tag{6.2}$$

In writing eq. (6.7), we implicitly take l_j to be the independent loop momentum and use eq. (3.12) to determine the other l_i in $\big|\widetilde{\mathcal{G}}(G;l_1,\ldots,l_n;p_1,\ldots,p_m)\big\rangle$. In particular, $l_{j+1} = l_j - p_i$.

One can use the collinear coefficient function to remove the collinear divergence. Consider the integral

$$\begin{aligned} \int \frac{d^d l}{(2\pi)^d} \bigg\{ \big|\widetilde{\mathcal{G}}(G;\{l\},p_1,\ldots,p_m)\big\rangle - \frac{1}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)} \times \\ \times \int_0^1 dx\; \delta\!\left(x - \frac{l_j \cdot n_i}{p_i \cdot n_i}\right) \big|f_i^{C,0}(G;x;p_1,\ldots,p_m)\big\rangle \bigg\}\,. \end{aligned} \tag{6.3}$$

Here we define the momentum fraction x away from the collinear limit by using a lightlike vector n_i. A good choice for n_i is

$$n_i^\mu = -p_i^\mu + \frac{2\, p_i \!\cdot\! w}{w^2} w^\mu\,, \qquad w^\mu = \sum_{k \in \text{final state}} p_k^\mu\,. \tag{6.4}$$

By construction, the integral in eq. (6.3) does *not* have a divergence from the collinear region (6.1). The divergence was only logarithmic and the subtraction removes the leading singularity, leaving at worst an integrable singularity.

There are two problems with eq. (6.3). The first is that we have already subtracted something from the integrand for the graph G, so we do not start with a clean slate. In the present notation, the soft subtraction term corresponding to propagator j in the loop being soft is, in the collinear limit (6.1),

$$\frac{1}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)}\, \frac{1}{x} \lim_{y \to 0} y \big|f_i^{C,0}(G;y;p_1,\ldots,p_m)\big\rangle\,. \tag{6.5}$$

(If the parton in loop propagator j is not a gluon, then this quantity is zero.) Similarly the soft subtraction term corresponding to propagator $j+1$ in the loop being soft is, in the collinear limit (6.1),

$$\frac{1}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)}\, \frac{1}{1-x} \lim_{y \to 1} (1-y) \big|f_i^{C,0}(G;y;p_1,\ldots,p_m)\big\rangle\,. \tag{6.6}$$

In order to cancel these contributions to the net integrand in the collinear limit, we should subtract them from the collinear subtraction term in eq. (6.3). To this end, we define a revised collinear coefficient function $\big|f_i^C(G;x;p_1,\dots,p_m)\big\rangle$,

$$\begin{aligned}\big|f_i^C(G;x;p_1,\dots,p_m)\big\rangle &= \big|f_i^{C,0}(G;x;p_1,\dots,p_m)\big\rangle - \frac{1}{x}\lim_{y\to 0}\, y\big|f_i^{C,0}(G;y;p_1,\dots,p_m)\big\rangle - \\ &\quad - \frac{1}{1-x}\lim_{y\to 1}\,(1-y)\big|f_i^{C,0}(G;y;p_1,\dots,p_m)\big\rangle\,. \end{aligned}\tag{6.7}$$

With this coefficient function, our subtraction inside the loop integral is

$$\frac{1}{(l_j^2+\mathrm{i}0)((p_i-l_j)^2+\mathrm{i}0)}\int_0^1 dx\;\delta\!\left(x-\frac{l_j\cdot n_i}{p_i\cdot n_i}\right)\big|f_i^C(G;x;p_1,\dots,p_m)\big\rangle\,. \tag{6.8}$$

A second problem remains with eq. (6.8). The integral of the subtraction term is ultraviolet divergent. We can easily fix that by modifying the subtraction term to be

$$\frac{f_{UV}(l_j,l_j-p_i)}{(l_j^2+\mathrm{i}0)((p_i-l_j)^2+\mathrm{i}0)}\int_0^1 dx\;\delta\!\left(x-\frac{l_j\cdot n_i}{p_i\cdot n_i}\right)\big|f_i^C(G;x;p_1,\dots,p_m)\big\rangle\,, \tag{6.9}$$

where

$$f_{UV}(l_j,l_j-p_i) = \frac{1}{2}\left(\frac{-\mu^2 e}{l_j^2-\mu^2 e+\mathrm{i}0}+\frac{-\mu^2 e}{(l_j-p_i)^2-\mu^2 e+\mathrm{i}0}\right). \tag{6.10}$$

Here μ is the $\overline{\rm MS}$ renormalization scale and $e = 2.71828\dots$ is the base of natural logarithms. The factor f_{UV} provides an extra power of l_j^2 in the denominator for large l_j but equals 1 in the collinear limit.

In summary, we subtract from the integrand for each graph G collinear subtraction terms

$$\sum_{i\in I_C(G)}\big|\widetilde{\mathcal{C}}_i(G;\{l\},p_1,\dots,p_m)\big\rangle\,, \tag{6.11}$$

where

$$\begin{aligned}\big|\widetilde{\mathcal{C}}_i(G;\{l\},p_1,\dots,p_m)\big\rangle &= \frac{f_{UV}(l_j,l_j-p_i)}{(l_j^2+\mathrm{i}0)((p_i-l_j)^2+\mathrm{i}0)}\times \\ &\quad\times\int_0^1 dx\;\delta\!\left(x-\frac{l_j\cdot n_i}{p_i\cdot n_i}\right)\big|f_i^C(G;x;p_1,\dots,p_m)\big\rangle\,.\end{aligned}\tag{6.12}$$

These subtractions, together with the soft subtractions, remove all of the infrared divergences from the loop integrals. One should note that, although the definition, eq. (6.7), of the collinear subtraction terms seems rather complicated, it is a purely algebraic construction that can be accomplished by straightforward computer algebra.

Our next task, pursued in the following two subsections, will be to add back the subtraction terms $\big|\widetilde{\mathcal{C}}_i(G;\{l\},p_1,\dots,p_m)\big\rangle$, this time integrated and summed over graphs G. That is, we want to add back the sum over graphs of

$$\big|\mathcal{C}_i(G;p_1,\dots,p_m)\big\rangle \equiv \int\frac{d^dl}{(2\pi)^d}\;\big|\widetilde{\mathcal{C}}_i(G;\{l\},p_1,\dots,p_m)\big\rangle\,. \tag{6.13}$$

JHEP09(2003)055

6.1 Quark line

For each one-loop virtual graph G and for each of its external lines i with $i \in I_C(G)$, we have subtracted a quantity $\big|\mathcal{C}_i(G;p_1,\dots,p_m)\big\rangle$ from $\big|\mathcal{G}(G;p_1,\dots,p_m)\big\rangle$. These subtractions soften the collinear divergences in $\big|\mathcal{G}(G;p_1,\dots,p_m)\big\rangle$ with its soft gluon subtractions, so that one could perform the loop integral for graph G by numerical integration. Now we must add the collinear subtractions $\big|\mathcal{C}_i(G;p_1,\dots,p_m)\big\rangle$ back, this time performing the integral analytically. By design, these integrals have collinear singularities that give poles at $d=4$ dimensions. The idea is to cancel these poles against the poles from the subtraction terms for collinear divergences in NLO tree graphs.

Unfortunately, the structure of the $\big|\mathcal{C}_i(G;p_1,\dots,p_m)\big\rangle$ for any given graph G is quite complicated. Fortunately, the structure of the sum over graphs G of $\big|\mathcal{C}_i(G;p_1,\dots,p_m)\big\rangle$ is vastly simpler. One gets a singular factor times the leading order matrix element $\big|\mathcal{M}(p_1,\dots,p_m)\big\rangle$ calculated from tree graphs. To see this, we simply need to apply the construction used to prove factorization of collinear parton contributions in hadron-hadron collisions [20].

In this subsection, we consider the case that line i is a quark line. The case that i is an antiquark line follows trivially and is covered at the end of this section. The case that i is a gluon line is treated in the following subsection.

We begin by setting up a useful notation. By assumption, line i, carrying momentum p_i out of the graph, connects to a virtual loop in G. Line i must therefore connect to the loop at a quark-gluon-quark vertex. We choose to label the propagators in the loop so that the gluon line in the loop has label j and the quark line has label $j+1$. Thus a gluon line with label j carries momentum l_j into the vertex and the quark line with label $j+1$ carries momentum $-l_{j+1} = p_i - l_j$ into the vertex. We write the integral for the graph as

$$\begin{aligned}
\mathcal{G}(G,p_1,\dots,p_m)^\alpha &= \int \frac{d^d l}{(2\pi)^d}\, \widetilde{\mathcal{G}}(G;l_1,\dots,l_n,p_1,\dots,p_m)^\alpha \\
&= \mathrm{i} g_s \mu^\epsilon T^a_{\alpha\alpha'} \int \frac{d^d l_j}{(2\pi)^d} \frac{1}{(l_j^2+\mathrm{i}0)((p_i-l_j)^2+\mathrm{i}0)} \times \\
&\quad \times \bar u(p_i)\gamma_\mu(\not p_i - \not l_j)\big[\widetilde{\mathbf{V}}^\mu(G_i,p_1,\dots,p_i-l_j,\dots,p_m,l_j)\big]^{\,a}_{\,\alpha'}\,. \qquad (6.14)
\end{aligned}$$

Here we have abandoned our vector notation for spin and color space and written the spin and color indices that are needed for the calculation explicitly. We display a quark color index α for quark line i in $\mathcal{G}$, leaving the other indices on $\mathcal{G}$ unwritten. The amplitude $\widetilde{\mathbf{V}}$ has a color index a corresponding to the gluon j in the loop and a quark color index α' corresponding to the quark line $j+1$ in the loop. There is an explicit color matrix $T^a_{\alpha\alpha'}$ connecting the colors of $\widetilde{\mathbf{V}}$ to the color of $\mathcal{G}$. The amplitude $\widetilde{\mathbf{V}}$ carries a vector index μ corresponding to the polarization of the gluon line j. It also carries a Dirac spinor index for the quark line $j+1$. However, we use the matrix notation for the Dirac structure of this line without displaying the Dirac indices explicitly. The Dirac spinor $\bar u(p_i)$ represents the final state quark on line i. We have displayed the integration over the loop momentum l_j, the quark-gluon-quark vertex and the propagators for the quark and the gluon in the

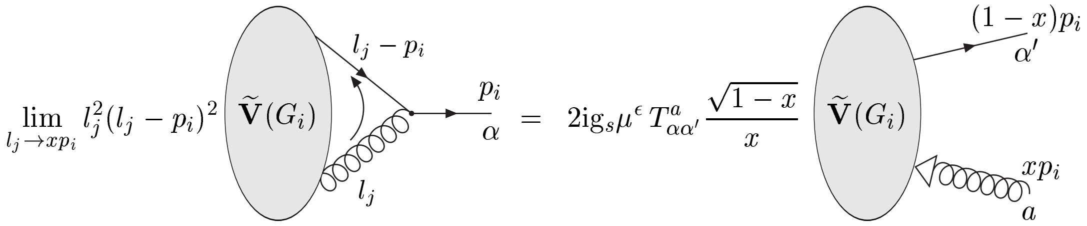

Figure 2: Illustration of eq. (6.15). The left hand side represents $f_i^{C,0}(G; x; p_1, \ldots, p_m)^\alpha$, which is defined in eq. (6.2) as the collinear limit of $l_j^2 (l_j - p_i)^2$ times the integrand $\widetilde{\mathcal{G}}(G; l_1, \ldots, l_n; p_1, \ldots, p_m)^\alpha$. Here the integrand $\widetilde{\mathcal{G}}$ consists of a spinor $\bar{u}(p_i)$, a vertex and two propagators times an amputated Green function $\widetilde{\mathbf{V}}$, as in eq. (6.14). The illustration represents the integrand: an integration over the loop momentum is *not* implied. On the right hand side, the gluon line of $\widetilde{\mathbf{V}}$ is contracted with $(xp_i)_\mu$, indicated by the arrowhead, while the quark line is contracted with $\bar{u}((1-x)p_i)$. The remaining factors in the right hand side of eq. (6.15) are indicated.

loop. Everything else is included in the Feynman amplitude $\widetilde{\mathbf{V}}$ for the amputated tree level graph G_i obtained by omitting the propagators j and $j+1$ and the vertex that attaches these propagators to external line i in graph G.

Thus the amplitude $\widetilde{\mathbf{V}}$ is defined by eq. (6.14). The momentum arguments of $\widetilde{\mathbf{V}}$ are the external momenta of the original graph, plus the momentum $p_i - l_j$ of the the quark line and the momentum l_j of the gluon line.

With the notation thus defined, we are ready to calculate. The subtraction term $f^{C,0}$ defined in eq. (6.2) can be expressed in terms of $\widetilde{\mathbf{V}}$ as

$$\begin{aligned} f_i^{C,0}(G; x; p_1, \ldots, p_m)^\alpha &= \lim_{l_j \to x\, p_i} l_j^2\, (p_i - l_j)^2\, \widetilde{\mathcal{G}}(G; l_1, \ldots, l_n; p_1, \ldots, p_m)^\alpha \\ &= \mathrm{i} g_s \mu^\epsilon T^a_{\alpha\alpha'} \bar{u}(p_i) \gamma_\mu (1-x) \not{p}_i \times \\ &\quad \times \left[\widetilde{\mathbf{V}}^\mu(G_i, p_1, \ldots, (1-x)p_i, \ldots, p_m, xp_i) \right]^a_{\alpha'} \\ &= 2 \mathrm{i} g_s \mu^\epsilon T^a_{\alpha\alpha'} \frac{\sqrt{1-x}}{x}\, (xp_i)_\mu\, \bar{u}((1-x)p_i) \times \\ &\quad \times \left[\widetilde{\mathbf{V}}^\mu(G_i, p_1, \ldots, (1-x)p_i, \ldots, p_m, xp_i) \right]^a_{\alpha'} . \end{aligned} \tag{6.15}$$

In the last step, we rewrite $\bar{u}(p_i)$ as $\bar{u}((1-x)p_i)/\sqrt{1-x}$ in order to have a spinor evaluated at the momentum of the corresponding quark line. This equation is illustrated in figure 2.

We would now like to sum over graphs. We consider one-loop graphs G with m external particles having flavors $\{f_1, \ldots, f_m\}$ and momenta $\{p_1, \ldots, p_m\}$. We are considering the collinear subtraction for parton i, which we assume here is a quark. Our graphs are amputated on their external legs. Furthermore, we need consider only graphs that have a collinear divergence for external leg i: $i \in I_C(G)$. Let us call this class of graphs C.

The corresponding graphs G_i are amputated tree graphs with $m+1$ external legs. The flavors of the first m external particles are $\{f_1, \ldots, f_m\}$, the same as for the graph G. The momenta of these particles are the same as for G except for particle i, which carries momentum $(1-x)p_i$ instead of p_i. The external particle with index $m+1$ is a gluon with momentum xp_i. Let us call the set of all such graphs C'. When we sum over all graphs

$G \in C$, the graphs G_i include all graphs in C' except for the graphs in which gluon $m+1$ couples directly to quark i. These graphs are absent because graphs with a self-energy insertion on external line i are *not* included in C. Let us call the set of $(m+1)$-particle graphs that we do get C'_+ and the graphs that we don't get C'_-.

The collinear subtraction defined in eq. (6.12), summed over graphs $G \in C$ is

$$
\begin{aligned}
\sum_{G\in C} \mathcal{C}_i(G;p_1,\dots,p_m)^\alpha &= \int \frac{d^d l_j}{(2\pi)^d} \frac{f_{UV}(l_j, l_j - p_i)}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)} \times \\
&\quad \times \int_0^1 dx\ \delta\biggl(x - \frac{l_j\cdot n_i}{p_i\cdot n_i}\biggr) \sum_{G\in C} f^C(G;x;p_1,\dots,p_m)^\alpha \\
&= \int \frac{d^d l_j}{(2\pi)^d} \frac{f_{UV}(l_j, l_j - p_i)}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)} \times \\
&\quad \times \int_0^1 \frac{dx}{x} \delta\biggl(x - \frac{l_j\cdot n_i}{p_i\cdot n_i}\biggr)\, 2\mathrm{i}g_s\mu^\epsilon\, T^a_{\alpha\alpha'} \times \\
&\quad \times \{(1-x)\, H^a_{\alpha'}(x;p_1,\dots,p_m) - H^a_{\alpha'}(0,p_1,\dots,p_m)\}\,. \qquad (6.16)
\end{aligned}
$$

Here we define

$$
\begin{aligned}
H^a_{\alpha'}(x;p_1,\dots,p_m) &= (1-x)^{-1/2} \lim_{q\to xp_i} \sum_{G_i\in C'_+} q_\mu\, \bar u((1-x)p_i) \times \\
&\quad \times \Bigl[\widetilde{\mathbf{V}}^\mu(G_i,p_1,\dots,(1-x)p_i,\dots,p_m,q)\Bigr]^a_{\alpha'}\,. \qquad (6.17)
\end{aligned}
$$

Note that we have written the momentum of parton $m+1$ as q instead of xp_i. Then we take the limit as $q \to xp_i$. This avoids an ambiguous expression of the form $0/0$ in the subsequent calculation.

We now note that gauge invariance says something about H. We have inserted a gluon line carrying momentum q almost everywhere into tree graphs with m legs. The gluon has polarization q^μ. Gauge invariance tells us that if we inserted this gluon everywhere, that is if we summed over the entire set of graphs C', we would get zero. However we have summed only over the graphs in C'_+, leaving out the graphs in C'_-. Thus H equals the negative of the sum over graphs in C'_-. This enables us to calculate H as follows:

$$
\begin{aligned}
H^a_{\alpha'}(x;p_1,\dots,p_m) &= -\lim_{q\to xp_i} \sum_{G_i\in C'_-} (1-x)^{-1/2} q_\mu\, \bar u((1-x)p_i) \times \\
&\quad \times \Bigl[\widetilde{\mathbf{V}}^\mu(G_i,p_1,\dots,(1-x)p_i,\dots,p_m,q)\Bigr]^a_{\alpha'} \\
&= \lim_{q\to xp_i} \sum_{G'_i\in C_0} (1-x)^{-1/2}\, g_s\mu^\epsilon T^a_{\alpha'\alpha''} \frac{\bar u((1-x)p_i)\not{q}((1-x)\not{p}_i + \not{q})}{((1-x)p_i + q)^2 + \mathrm{i}0} \times \\
&\quad \times \Bigl[\widetilde{\mathbf{W}}(G'_i,p_1,\dots,(1-x)p_i + q,\dots,p_m)\Bigr]_{\alpha''} \\
&= \sum_{G'_i\in C_0} g_s\mu^\epsilon T^a_{\alpha'\alpha''}\, \bar u(p_i)\ \Bigl[\widetilde{\mathbf{W}}(G'_i,p_1,\dots,p_i,\dots,p_m)\Bigr]_{\alpha''} \\
&= g_s\mu^\epsilon T^a_{\alpha'\alpha''}\, \mathcal{M}(p_1,\dots,p_i,\dots,p_m)^{\alpha''}\,. \qquad (6.18)
\end{aligned}
$$

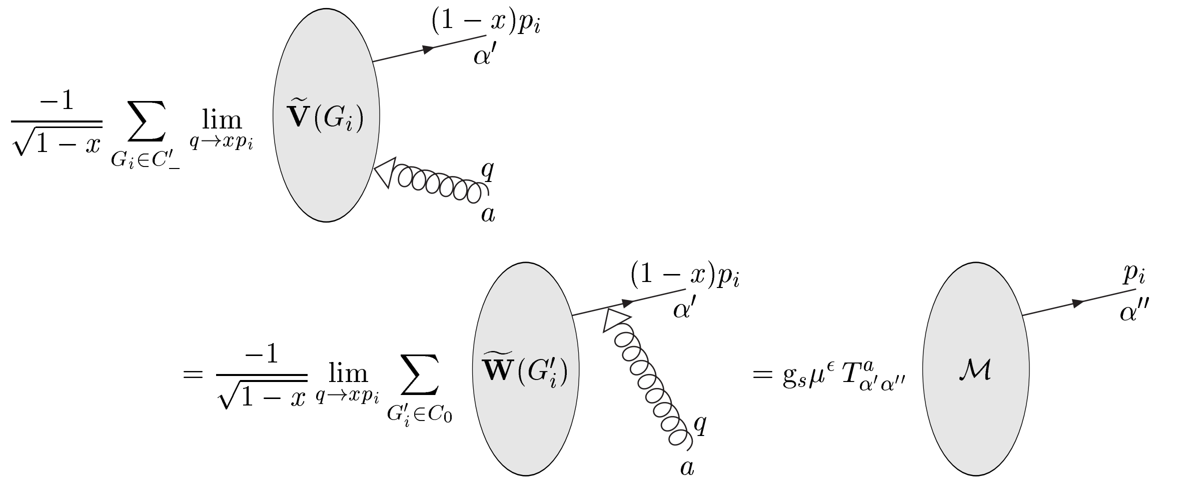

Figure 3: Illustration of eq. (6.18). The left hand side represents $H^a_{\alpha'}(x;p_1,\ldots,p_m)$ expressed in terms of $\widetilde{\mathbf{V}}$, with the gluon line contracted with q indicated by the curly line with an arrowhead. The sum is originally over graphs $G_i \in C'_+$, but this is the negative of the sum over graphs $G_i \in C'_-$, in which the gluon connects to line i, as depicted in the right hand side of the first equality. The remaining Green function is $\widetilde{\mathbf{W}}$. The q^μ insertion then cancels the adjoining propagator. After summing over graphs, we are left with a color matrix times the complete m particle tree amplitude.

Following the second equals sign, we have displayed the vertex at which the gluon couples to the quark line i, together with the adjacent propagator. Everything else we call $\widetilde{\mathbf{W}}$. Note that $\widetilde{\mathbf{W}}$ is the Green function for an amputated tree graph G'_i with m external partons. The partons have flavors $\{f_1,\ldots,f_m\}$, just as with our original loop graphs, and they have the same momenta as the original partons except that parton i carries momentum $(1-x)p_i + q$. Let us denote the set of all such graphs C_0. The sum over $G_i \in C'_-$ implies that G'_i runs over all of C_0. In the next step, we replace $\not{q}$ by $((1-x)\not{p}_i + \not{q})$ next to the spinor. This gives a factor $((1-x)p_i + q)^2/((1-x)p_i+q)^2 = 1$. After this cancellation, it is safe to take the limit $q \to xp_i$. Also, we replace $\bar{u}((1-x)p_i)$ by $\sqrt{1-x}\,\bar{u}(p_i)$. In the last step, we recognize that we have the complete three level amplitude $\mathcal{M}$ for the m external particles. This calculation is illustrated in figure 3.

We can now insert this result into eq. (6.16) and perform the loop integral to obtain

$$
\begin{aligned}
\sum_{G, i \in I_C(G)} \mathcal{C}_i(G;p_1,\ldots,p_m)^\alpha &= \int \frac{d^d l_j}{(2\pi)^d} \frac{f_{UV}(l_j, l_j - p_i)}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)} \times \\
&\quad \times \int_0^1 \frac{dx}{x}\, \delta\!\left(x - \frac{l_j \cdot n_i}{p_i \cdot n_i}\right) 2\mathrm{i}\mathrm{g}_s \mu^\epsilon \, T^a_{\alpha\alpha'} \times \\
&\quad \times \left\{ [(1-x)-1] \mathrm{g}_s \mu^\epsilon T^a_{\alpha'\alpha''} \, \mathcal{M}(p_1,\ldots,p_i,\ldots,p_m)^{\alpha''} \right\} \\
&= -2\mathrm{i}\mathrm{g}_s^2 C_F \, \mu^{2\epsilon} \int \frac{d^d l_j}{(2\pi)^d} \frac{f_{UV}(l_j, l_j - p_i)}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)} \times \\
&\quad \times \mathcal{M}(p_1,\ldots,p_i,\ldots,p_m)^\alpha
\end{aligned}
$$

JHEP09(2003)055

$$= \frac{\alpha_s}{4\pi} C_F \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left(-\frac{2}{\epsilon} + \mathcal{O}(\epsilon) \right) \times$$
$$\times \mathcal{M}(p_1, \ldots, p_i, \ldots, p_m)^\alpha \,. \tag{6.19}$$

Thus, when we sum over all graphs the collinear subtraction terms associated with an external quark line with label i, we get a simple singular factor times the tree level amplitude $\mathcal{M}$.

For an external antiquark line we get the same result from essentially the same calculation. We just have to exchange $u(p_i)$ for $\bar{v}(p_i)$, reverse the order of matrix multiplications in Dirac spinor space, and replace $\not{k} \to -\not{k}$ in the Dirac propagator numerators.

6.2 Gluon line

In this subsection we calculate the sum over graphs of the collinear subtractions $\big|\mathcal{C}_i(G)\big\rangle$ in the case that the external line i is a gluon line. There are two possibilities. Either the external gluon connects to a virtual quark and antiquark, or else it connects to two virtual gluons.

The case in which the external gluon connects to a virtual quark and antiquark is simple. In this case the collinear limit of the numerator is proportional to

$$x(1-x)\not{p}_i \not{\varepsilon}(p_i) \not{p}_i = x(1-x) \left[-p_i^2 \not{\varepsilon}(p_i) + 2\not{p}_i \, p_i \cdot \varepsilon(p_i) \right] = 0 \,, \tag{6.20}$$

where $\varepsilon^\mu(p)$ is the polarization vector of the external gluon. The first term is zero because of the $p_i^2 = 0$. The second term is also zero because we consider only physical polarizations of the external gluon, so that $p_i \cdot \varepsilon(p_i) = 0$. Thus $\big|\mathcal{C}_i(G; p_1, \ldots, p_m)\big\rangle$ is zero when the external gluon connects to a virtual quark and antiquark.

We now turn to the case in which the external gluon connects to two virtual gluons, $g + g \to g$. Our analysis is similar to that of the previous subsection for $q + g \to q$. There are, however, some subtleties, so we present the argument in some detail.

By assumption, line i, carrying momentum p_i out of the graph, connects to a virtual loop in G at a gluon-gluon-gluon vertex. We suppose that the gluon lines in the loop have labels j and $j+1$. Thus the gluon line with label j carries momentum l_j into the vertex and the gluon line with label $j+1$ carries momentum $-l_{j+1} = p_i - l_j$ into the vertex. We write the integral for the graph as

$$\begin{aligned}
\mathcal{G}(G, p_1, \ldots, p_m)_b &= \int \frac{d^d l}{(2\pi)^d} \, \widetilde{\mathcal{G}}(G; l_1, \ldots, l_n, p_1, \ldots, p_m)_b \\
&= -\mathrm{i} g_s \mu^\epsilon T^a_{bc} \int \frac{d^d l_j}{(2\pi)^d} \, \frac{\varepsilon^\sigma(p_i) V^{(3)}_{\sigma\nu\mu}(p_i, l_j - p_i, -l_j)}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)} \times \\
&\quad \times \left[\widetilde{\mathbf{V}}(G_i, p_1, \ldots, p_i - l_j, \ldots, p_m, l_j) \right]^{\mu\nu}_{ac} .
\end{aligned} \tag{6.21}$$

We display a gluon color index b for line i in $\mathcal{G}$, leaving the other indices on $\mathcal{G}$ unwritten. The amplitude $\widetilde{\mathbf{V}}$ has a color index a corresponding to the gluon j in the loop and a gluon color index c corresponding to line $j+1$ in the loop. There is an explicit color matrix $T^a_{bc} = i f_{bac}$ connecting the colors of $\widetilde{\mathbf{V}}$ to the color of $\mathcal{G}$. The amplitude $\widetilde{\mathbf{V}}$ also carries vector indices μ, ν corresponding to the polarization of the gluon lines $j, j+1$. The polarization vector $\epsilon^\sigma(p_i)$ represents the final state gluon on line i. We have displayed the integration over the loop momentum l_j, the propagators for the gluons in the loop and the gluon-gluon-gluon vertex V,

$$V_{\sigma\nu\mu}(p_i, l_j - p_i, -l_j) = g_{\sigma\nu}(l_j - 2p_i)_\mu + g_{\nu\mu}(p_i - 2l_j)_\sigma + g_{\mu\sigma}(p_i + l_j)_\nu\,. \tag{6.22}$$

Everything else is included in the Feynman amplitude $\widetilde{\mathbf{V}}$ for the amputated tree level graph G_i obtained by omitting the propagators j and $j+1$ and the vertex that attaches these propagators to external line i in graph G. Thus the amplitude $\widetilde{\mathbf{V}}$ is defined by eq. (6.21). The momentum arguments of $\widetilde{\mathbf{V}}$ are the external momenta of the original graph, plus the momenta $p_i - l_j$ and l_j of the two gluon lines.

With the notation thus defined, we are ready to calculate. The subtraction term $f^{C,0}$ defined in eq. (6.2) can be expressed in terms of $\widetilde{\mathbf{V}}$ as

$$\begin{aligned} f_i^{C,0}(G;x;p_1,\dots,p_m)_b &= \lim_{l_j \to x\, p_i} l_j^2\,(p_i - l_j)^2\,\widetilde{\mathcal{G}}(G;l_1,\dots,l_n;p_1,\dots,p_m)_b \\ &= \lim_{l_j \to x\, p_i} (-\mathrm{i}g_s\mu^\epsilon)T^a_{bc}\varepsilon^\sigma(p_i)\,V_{\sigma\nu\mu}(p_i, l_j - p_i, -l_j)\,\times \\ &\qquad \times\Big[\widetilde{\mathbf{V}}(G_i,p_1,\dots,p_i - l_j,\dots,p_m,l_j)\Big]^{\mu\nu}_{ac} \\ &= \mathrm{i}g_s\mu^\epsilon T^a_{bc}\frac{2-x}{x}\,(xp_i)_\mu\,\varepsilon_\nu((1-x)p_i)\,\times \\ &\qquad \times\Big[\widetilde{\mathbf{V}}(G_i,p_1,\dots,(1-x)p_i,\dots,p_m,xp_i)\Big]^{\mu\nu}_{ac} + \\ &\quad + \mathrm{i}g_s\mu^\epsilon T^c_{ba}\frac{1+x}{1-x}((1-x)p_i)_\nu\,\varepsilon_\mu(xp_i)\,\times \\ &\qquad \times\Big[\widetilde{\mathbf{V}}(G_i,p_1,\dots,(1-x)p_i,\dots,p_m,xp_i)\Big]^{\mu\nu}_{ac}\,. \end{aligned} \tag{6.23}$$

There are two terms here. In the first, we contract the polarization index ν of the gluon line with momentum $(1-x)p_i$ into the index ν of the polarization vector $\varepsilon_\nu(p_i) = \varepsilon_\nu((1-x)p_i)$, while we contract the polarization index μ of the gluon line with momentum xp_i with the index μ of its momentum vector $(xp_i)_\mu$. In the second term the situation is just reversed, so that gluon line with index ν and momentum $(1-x)p_i$ is contracted with its momentum vector. It is useful to represent these two terms diagrammatically as different graphs, depicting the gluon line contracted with the polarization vector as a curly line and the gluon line contracted with its momentum vector as curly line with an arrowhead, as illustrated in figure 4. We can denote these two graphs as G_i^1 and G_i^2.

In the case of the second term, we revise the notation, exchanging $\mu \leftrightarrow \nu$, $a \leftrightarrow c$, $x \leftrightarrow (1-x)$ and exchanging the positions of the xp_i and $(1-x)p_i$ arguments of $\widetilde{\mathbf{V}}$. Then the two terms have the same form except that in one term we have the amplitude $\widetilde{\mathbf{V}}(G_i^1,\dots)$ and in the other we have $\widetilde{\mathbf{V}}(G_i^2,\dots)$

As in the previous subsection, we would now like to sum over graphs. We consider one-loop amputated graphs G with m external particles having flavors $\{f_1,\dots,f_m\}$ and momenta $\{p_1,\dots,p_m\}$. We are considering the collinear subtraction for parton i, which we assume here is a gluon. We consider only graphs that have a collinear divergence for external leg i: $i \in I_C(G)$. We call this class of graphs C.

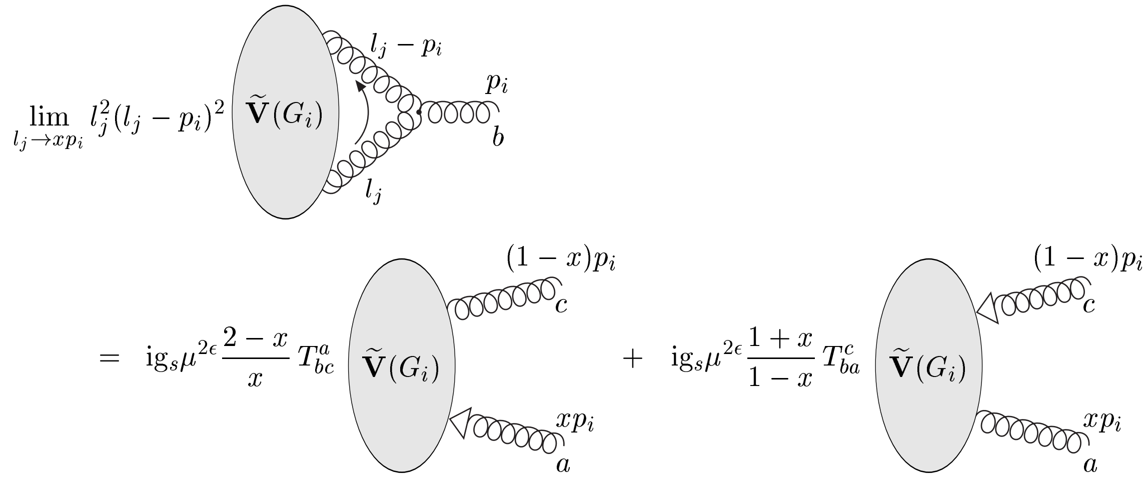

Figure 4: Illustration of eq. (6.23). The left hand side represents $f_i^{C,0}(G; x; p_1, \dots, p_m)^\alpha$, which is defined in eq. (6.2) as the collinear limit of $l_j^2\,(l_j - p_i)^2$ times the integrand $\widetilde{\mathcal{G}}(G; l_1, \dots, l_n; p_1, \dots, p_m)^\alpha$. Here the integrand $\widetilde{\mathcal{G}}$ consists of a polarization vector $\varepsilon^\sigma(p_i)$, a vertex and two propagators times an amputated Green function $\widetilde{\mathbf{V}}$, as in eq. (6.21). As in figure 2, the illustration represents the integrand: an integration over the loop momentum is *not* implied. On the right hand side, there are two terms. In each, one gluon line of $\widetilde{\mathbf{V}}$ is contracted with its momentum vector (xp_i) or $(1-x)p_i$ respectively, indicated by the arrowhead. The other gluon line is contracted with the original polarization vector $\varepsilon^\sigma(p_i)$, which is written as $\varepsilon_\nu((1-x)p_i)$ in the first term and $\varepsilon_\mu(xp_i)$ in the first term. The remaining factors in the right hand side of eq. (6.23) are indicated.

The corresponding graphs G_i^1 and G_i^2 are amputated tree graphs with $m+1$ external legs. The flavors of the first m external particles are $\{f_1, \dots, f_m\}$, the same as for the graph G. The momenta of these particles are the same as for G except for particle i, which carries momentum $(1-x)p_i$ instead of p_i. The external particle with index $m+1$ is a gluon with momentum xp_i. Let us call the set of all such graphs C'. When we sum over all graphs $G \in C$, the graphs G_i^1 together with the graphs G_i^2 include all graphs in C' except for the graphs in which gluon $m+1$ couples directly to gluon i. These graphs are absent because graphs with a self-energy insertion on external line i are *not* included in C. Let us call the set of $(m+1)$-particle graphs that we do get C'_+ and the graphs that we don't get C'_-.

The collinear subtraction defined in eq. (6.12), summed over graphs $G \in C$, is

$$
\begin{aligned}
\sum_{G \in C} \mathcal{C}_i(G; p_1, \dots, p_m)_b = {} & \int \frac{d^d l_j}{(2\pi)^d} \frac{f_{UV}(l_j, l_j - p_i)}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)} \times \\
& \times \int_0^1 dx\; \delta\!\left(x - \frac{l_j \cdot n_i}{p_i \cdot n_i}\right) \sum_{G \in C} f_i^C(G; x; p_1, \dots, p_m)_b \\
= {} & \int \frac{d^d l_j}{(2\pi)^d}\; \frac{f_{UV}(l_j, l_j - p_i)}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)} \times \\
& \times \int_0^1 dx\, \delta\!\left(x - \frac{l_j \cdot n_i}{p_i \cdot n_i}\right) \mathrm{i} g_s \mu^\epsilon T^a_{bc} \times
\end{aligned}
$$

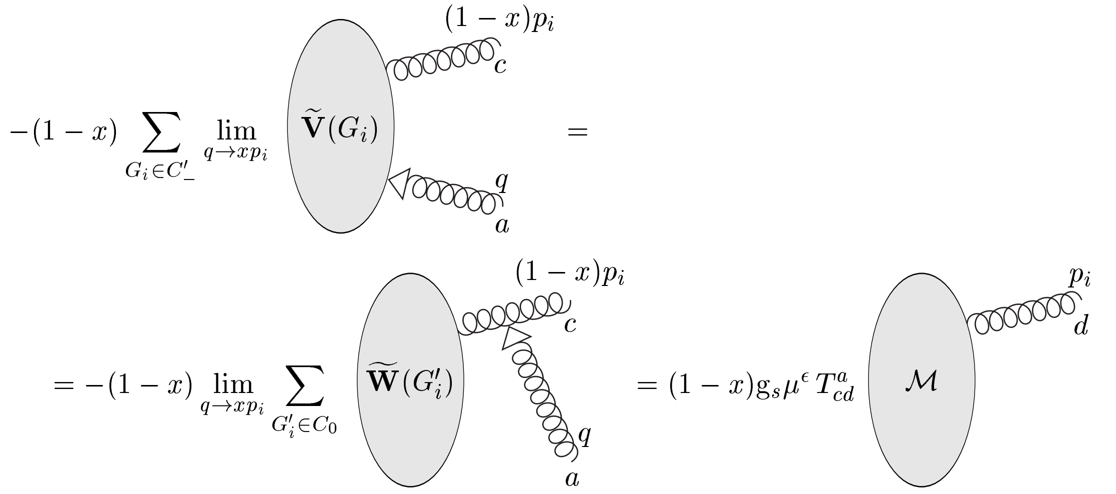

Figure 5: Illustration of eq. (6.26). The left hand side represents $H_{ac}(x;p_1,\ldots,p_m)$ expressed in terms of $\widetilde{\mathbf{V}}$, with the gluon line contracted with $q_{\rm i}$ indicated by the curly line with an arrowhead. The sum is originally over graphs $G_i \in C'_+$, but this is the negative of the sum over graphs $G_i \in C'_-$, in which the gluon connects to line i, as depicted in the right hand side of the first equality. The remaining Green function is $\widetilde{\mathbf{W}}$. The q_μ insertion then gives two terms, not depicted in the figure. One of the terms cancels when summed over graphs, while in the other the propagator adjoining the q_μ insertion is cancelled. After summing over graphs, we are left with a color matrix times the complete m particle tree amplitude.

$$
\begin{aligned}
&\times\left\{\frac{(2-x)}{x(1-x)}\,H_{ac}(x;p_1,\ldots,p_m) - \frac{2}{x}\,H_{ac}(0,p_1,\ldots,p_m) - \right.\\
&\left. - \frac{H_{ac}(1,p_1,\ldots,p_m)}{(1-x)}\right\}, \qquad (6.24)
\end{aligned}
$$

where we define

$$
\begin{aligned}
H_{ac}(x;p_1,\ldots,p_m) = {}& (1-x)\lim_{q\to xp_i}\sum_{G_i\in C'_+} q_\mu\,\varepsilon_\nu((1-x)p_i)\times\\
&\times\left[\widetilde{\mathbf{V}}(G_i,p_1,\ldots,(1-x)p_i,\ldots,p_m,q)\right]^{\mu\nu}_{ac}. \qquad (6.25)
\end{aligned}
$$

We now use gauge invariance. We have inserted a gluon line carrying momentum q with polarization q^μ almost everywhere into tree graphs with m legs. Gauge invariance tells us that if we inserted this gluon everywhere, that is if we summed over the entire set of graphs C', we would get zero. However we have summed only over the graphs in C'_+, leaving out the graphs in C'_-. Thus H equals the negative of the sum over graphs in C'_-. This enables us to calculate H as follows:

$$
\begin{aligned}
H_{ac}(x;p_1,\ldots,p_m) = {}& -(1-x)\lim_{q\to xp_i}\sum_{G_i\in C'_-} q_\mu\,\varepsilon_\nu((1-x)p_i)\times\\
&\times\left[\widetilde{\mathbf{V}}(G_i,p_1,\ldots,(1-x)p_i,\ldots,p_m,q)\right]^{\mu\nu}_{ac}\\
= {}& -(1-x)\lim_{q\to xp_i}\sum_{G'_i\in C_0} \mathrm{g}_s\mu^\epsilon T^a_{cd}\times
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{q_\mu \varepsilon_\nu((1-x)p_i) V^{\nu\lambda\mu}((1-x)p_i, -(1-x)p_i - q, q)}{((1-x)p_i+q)^2 + \mathrm{i}0} \times \\
&\times \left[\widetilde{\mathbf{W}}(G'_i, p_1, \dots, (1-x)p_i + q, \dots, p_m)\right]_{\lambda,d} \\
= {}& (1-x) \lim_{q\to xp_i} \mathrm{g}_s \mu^\epsilon \times \\
&\times \sum_{G'_i\in C_0} \Bigg\{ T^a_{cd}\, \varepsilon_\nu((1-x)p_i) \times \\
&\qquad\qquad \times \left[\widetilde{\mathbf{W}}(G'_i, p_1, \dots, (1-x)p_i + q, \dots, p_m)\right]^\nu_d - \\
&\qquad\qquad - T^a_{cd} \frac{q\cdot\varepsilon((1-x)p_i)\, ((1-x)p_i+q)_\lambda}{((1-x)p_i+q)^2 + \mathrm{i}0} \times \\
&\qquad\qquad \times \left[\widetilde{\mathbf{W}}(G'_i, p_1, \dots, (1-x)p_i + q, \dots, p_m)\right]^\lambda_d \Bigg\} \\
= {}& (1-x)\ \mathrm{g}_s \mu^\epsilon T^a_{cd}\, \varepsilon_\nu(p_i) \sum_{G'_i\in C_0} \left[\widetilde{\mathbf{W}}(G'_i, p_1, \dots, p_i, \dots, p_m)\right]^\nu_d \\
= {}& (1-x) \mathrm{g}_s \mu^\epsilon T^a_{cd}\, \mathcal{M}(p_1, \dots, p_i, \dots, p_m)^d \,. \qquad (6.26)
\end{aligned}
$$

After the second equals sign, we have displayed the vertex at which the gluon with momentum q couples to the gluon line i, together with the adjacent propagator. This multiplies the Green function $\widetilde{\mathbf{W}}$ for an amputated tree graph G'_i with m external partons. The partons have the same flavors as in our original loop graphs, and they have the same momenta except that parton i carries momentum $(1-x)p_i + q$. We denote the set of all such graphs C_0. The sum over $G_i \in C'_-$ implies that G'_i runs over all of C_0. The vector and color indices displayed for $\widetilde{\mathbf{W}}$ are those of the line i. Now, when we evaluate q_μ contracted with the vertex function and use $(1-x)p_i \cdot \varepsilon((1-x)p_i) = 0$ as well as $p_i^2 = 0$, we get two terms, as indicated on the right hand side of the third equality. In the second term, the amplitude $\widetilde{\mathbf{W}}$ is contracted with the momentum carried by the gluon line. After summing over graphs G'_i, this term vanishes. Finally, we take the limit $q \to xp_i$ and recognize that we have the complete three level amplitude $\mathcal{M}$ for m external particles. This calculation is illustrated in figure 5.

We can now insert this result into eq. (6.24) and perform the loop integral to obtain

$$
\begin{aligned}
\sum_{G\in C} \mathcal{C}_i(G; p_1, \dots, p_m)^b = {}& \int \frac{d^d l_j}{(2\pi)^d} \frac{f_{UV}(l_j, l_j - p_i)}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)} \int_0^1 dx\, \delta\!\left(x - \frac{l_j\cdot n_i}{p_i \cdot n_i}\right) \mathrm{i}\mathrm{g}_s\, \mu^\epsilon T^a_{bc} \times \\
&\times \left\{ \left[\frac{(2-x)}{x} - \frac{2}{x} - \frac{0}{1-x}\right] \mathrm{g}_s \mu^\epsilon T^a_{cd}\, \mathcal{M}(p_1, \dots, p_i, \dots, p_m)^d \right\} \\
= {}& -\mathrm{i}\mathrm{g}_s^2 C_A\, \mu^{2\epsilon} \int \frac{d^d l_j}{(2\pi)^d} \frac{f_{UV}(l_j, l_j - p_i)}{(l_j^2 + \mathrm{i}0)((p_i - l_j)^2 + \mathrm{i}0)} \mathcal{M}(p_1, \dots, p_i, \dots, p_m)^b \\
= {}& \frac{\alpha_s}{4\pi} C_A \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left(-\frac{1}{\epsilon} + \mathcal{O}(\epsilon)\right) \mathcal{M}(p_1, \dots, p_i, \dots, p_m)^b \,. \qquad (6.27)
\end{aligned}
$$

Thus, when we sum over all graphs the collinear subtraction terms associated with an external gluon line with label i, we get a simple singular factor times the tree level amplitude $\mathcal{M}$.

7. Final formulas

There are two steps to the algorithm that we have outlined here for generating the expressions to be used in a numerical calculation of the one-loop graphs for a QCD amplitude.

The first step applies graph by graph, generating subtractions for each graph. If the graph is ultraviolet divergent, one generates a subtraction term for the integrand of the graph. The subtraction term matches the integrand when the loop momentum is much larger than the external momenta. In this way, the ultraviolet divergence of the integral is removed. The subtraction terms are designed to have the same effect as standard $\overline{\text{MS}}$ renormalization. If the graph has soft or collinear divergences, or both, then one generates corresponding subtraction terms. The subtraction terms match the integrand in the soft and collinear limits of the loop momentum. In this way, the soft and collinear divergences of the integral are removed.

The second step generates a next-to-leading order contribution to the m-parton amplitude that is proportional to the tree level matrix element (summed over tree graphs). This contribution has the form

$$\mathrm{I}^V(\epsilon)\big|\mathcal{M}_{\text{tree}}\,(p_1,\dots,p_m)\,\big\rangle\,. \tag{7.1}$$

where $\mathrm{I}^V(\epsilon)$ is a singular function of the dimensional regularization parameter ϵ and is a linear operator on the color space of the amplitude $\big|\mathcal{M}_{\text{tree}}\,(p_1,\dots,p_m)\,\big\rangle$. There are three parts to $\mathrm{I}^V(\epsilon)$. The first comes from the factors $\sqrt{r_i}$ for each external line that relate the scattering matrix to the amputated Green function. These contributions are given in eq. (3.19). There is one term for each external line i. The second contribution comes from adding back the collinear subtractions. Again there is one contribution for each external line, as given in eq. (6.19) for quarks and eq. (6.27) for gluons. Finally, there is a third contribution that comes from adding back the soft subtractions. There is one term for each pair of external lines, as given in eq. (5.15). The net result is

$$\mathrm{I}^V(\epsilon) = \frac{\alpha_s}{4\pi}\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon^2}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\mathbf{T}_i\cdot\mathbf{T}_j\left(\frac{\mu^2}{-2p_i\!\cdot\! p_j}\right)^{\epsilon} - \frac{1}{\epsilon}\sum_{i=1}^{m}\gamma_i\right), \tag{7.2}$$

where the γ_i factors are

$$\gamma_q = \gamma_{\bar q} = \frac{3}{2}C_F\,, \qquad \gamma_g = \frac{11}{6}C_A - \frac{4}{6}T_R n_f\,. \tag{7.3}$$

The $1/\epsilon^2$ and $1/\epsilon$ poles are all of infrared origin.

We now turn to the complete calculation of an infrared safe cross section σ, which we write in the form (following as much as possible the notation of [15])

$$\sigma = \sigma^{\text{LO}\{m-n_I\}} + \sigma^{\text{NLO}\{m+1-n_I\}} + \sigma^{\text{NLO}\{m-n_I\}}\,. \tag{7.4}$$

There are a lowest order term and two next-to-leading order terms. We suppose that there are n_I initial state particles. (In applications, n_I is 0, 1, or 2.) In the LO term there are m total particles, n_I in the initial state and the remaining $m-n_I$ in the final state. We denote this contribution by $\sigma^{\mathrm{LO}\{m-n_I\}}$, but do not discuss its calculation. The NLO term from real parton emission has $m+1-n_I$ final state particles. This is calculated from the $m+1-n_I$ particle matrix element minus subtractions. One can use, for example, the subtraction scheme proposed by Catani and Seymour [15]. We denote this contribution by $\sigma^{\mathrm{NLO}\{m+1-n_I\}}$, but do not further discuss its calculation. The remaining NLO term has $m-n_I$ final state particles. It is calculated as follows:

$$
\begin{aligned}
\sigma^{\mathrm{NLO}\{m-n_I\}} = & \sum_G \int d\Phi^{m-n_I}\; F_J^{(m-n_I)}(p_1,\dots,p_m) \int \frac{d^4 l}{(2\pi)^4} \times \\
& \times 2\,\mathrm{Re}\Big\{ \big\langle \mathcal{M}_{\mathrm{tree}}(p_1,\dots,p_m) \big| \widetilde{\mathcal{G}}(G;\{l\};p_1,\dots,p_m) \big\rangle - \\
& \quad - \big\langle \mathcal{M}_{\mathrm{tree}}(p_1,\dots,p_m) \big| \widetilde{\mathcal{R}}(G;\{l\};p_1,\dots,p_m) \big\rangle - \\
& \quad - \sum_{\{i,j\}\in I_S(G)} \big\langle \mathcal{M}_{\mathrm{tree}}(p_1,\dots,p_m) \big| \widetilde{\mathcal{S}}_{ij}(G;\{l\};p_1,\dots,p_m) \big\rangle - \\
& \quad - \sum_{i\in I_C(G)} \big\langle \mathcal{M}_{\mathrm{tree}}(p_1,\dots,p_m) \big| \widetilde{\mathcal{C}}_i(G;\{l\};p_1,\dots,p_m) \big\rangle \Big\} + \\
& + \int d\Phi^{m-n_I}\; F_J^{(m-n_I)}(p_1,\dots,p_m) \\
& \quad \times 2 \lim_{\epsilon\to 0} \big\langle \mathcal{M}_{\mathrm{tree}}(p_1,\dots,p_m) \big| \mathrm{I}^V(\epsilon) + \mathrm{I}^R(\epsilon) \big| \mathcal{M}_{\mathrm{tree}}(p_1,\dots,p_m) \big\rangle\,. \qquad (7.5)
\end{aligned}
$$

In the first term here, there is a sum over amputated one-loop graphs G. We integrate over the phase space $d\Phi$ for $m-n_I$ final state particles with momenta $p_{n_I+1},\dots p_m$, where the initial state particles have momenta $p_1,\dots,p_{n_I}$. Next, we supply a measurement F_J appropriate to the infrared safe observable that we wish to calculate. ("J" is for "jet".) Then there is an integration over the loop momentum l as described in section 3.2. Now there follow four terms inside the loop integration. The first is the integrand $\widetilde{\mathcal{G}}$ for the graph G times the complex conjugate of the tree amplitude for $m-n_I$ final state particles. The second is $\langle \mathcal{M}_{\mathrm{tree}}|$ times the renormalization subtraction $\mathcal{R}$, which is zero if the virtual loop graph is ultraviolet convergent. The renormalization subtraction is defined in section 4. The third term contains the soft gluon subtractions $\mathcal{S}$ for graph G, one subtraction term for each pair $\{i,j\}$ of external lines that corresponds to a soft divergence. The soft subtraction is defined in section 5. The fourth term contains the collinear subtractions $\mathcal{C}$ for graph G, one subtraction term for each external line i that corresponds to a collinear divergence. The collinear subtraction is defined in section 6.

There is one final term, which has no integral over a loop momentum and involves only the tree amplitude. Here we have the singular function $\mathrm{I}^V(\epsilon)$ from eq. (7.2) that adds back the soft and collinear subtractions from the virtual graphs. We also have a singular function $\mathrm{I}^R(\epsilon)$ that adds back the soft and collinear subtractions from the real graphs. Naturally, exactly what function goes here depends on what subtraction scheme one uses for the real graphs. However, the $1/\epsilon$ and $1/\epsilon^2$ poles, whose structure follows from the structure of QCD, cancel between $\mathrm{I}^V(\epsilon)$ and $\mathrm{I}^R(\epsilon)$. (For instance, one can easily check that this cancellation works for the scheme of [15], where $\mathbf{I}(\epsilon)$ corresponds to our $2\,\mathbf{I}^R(\epsilon)$.) Generally a finite contribution remains in the limit $\epsilon \to 0$.

JHEP09(2003)055

8. Prospects

In this paper, we have analyzed the NLO contribution to a QCD cross section from graphs with virtual loops. We have seen how to represent these contributions in the form of simple terms with poles in $d-4$ plus suitably subtracted integrals over the momenta of the final state particles and over the momentum l in the loop. The subtractions are such that the integration over l is convergent in $d=4$ dimensions. The proposal is to perform the subtracted integrals numerically.

We are motivated by the idea that any integral that is finite can be performed by numerical integration. However, it can be a challenge to put this idea into practice.

The first challenge lies with the singularities in the integrand of the form

$$\frac{\mathrm{i}}{(l-K)^2+\mathrm{i}0}\,. \tag{8.1}$$

The "+i0" instruction here is telling us to deform the integration contour in order to avoid the pole. That is, we should replace l^μ by $l^\mu + \mathrm{i}\kappa^\mu$ where κ^μ is real and depends on l^μ. Then $(l+\mathrm{i}\kappa-K)^2$ has an imaginary part $2(l-K)\cdot\kappa$ and the "+i0" instructs us to make sure that $2(l-K)\cdot\kappa > 0$ in the integration region in which $(l-K)^2$ is small. This is what we need for most of the integration range, but there is an exception near certain "pinch singularities" at which κ^μ, or at least the parts of κ^μ in certain directions, must vanish. The pinch singularities include, most notably, the collinear and soft singularities for which we have subtractions. Thus one needs a construction of a suitable function $\kappa^\mu(l)$. There is a lot of flexibility in choosing κ, but one wants a choice that is general and simple. The methods of [8]–[11] can be useful here, but these methods apply for up to four lines in the loop while we need a method that works for any number of lines in the loop. Fortunately, the needed method will be for four dimensions, in which the structure of singularities is very simple, whereas the construction of [8]–[11] was for a three-dimensional case in which the singularity structure was more complicated.

The second challenge in putting numerical integration into practice is to develop a suitable sampling method for choosing the points l in a Monte Carlo style integration. The density of points needs to be suitably singular at each of the pinch singularities of the integrand. This problem is quite analogous to the problem of choosing integration points for NLO graphs that do *not* have a virtual loop.

The third challenge is to generate the sum over virtual graphs G in eq. 7.5. In a method in which the virtual loop integrals are performed analytically, one can also sum over graphs G analytically, but here we need to implement the sum over graphs numerically.

We expect that these challenges can be met.

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A. UV counterterms

In this appendix we give the ultraviolet counterterms for the one-loop propagator and vertex corrections in the massless case. We follow the general prescription given in section 4, which works in every case except for the gluon self-energy, where we need an extra subtraction that was not discussed in section 4 in order to handle the quadratic divergence.

Since the divergent graphs and the corresponding counterterms depend on the form of the Feynman rules it is useful to define them. In Feynman gauge we have the following rules:

- Gluon propagator:

$$\mathrm{i}D^{\mu\nu}_{ab}(k) = -\frac{\mathrm{i}g_{\mu\nu}}{k^2+\mathrm{i}0}\,\delta_{ab}\,, \tag{A.1}$$

  where a and b are the color indices and k is the momentum carried by this line.

- Quark propagator:

$$\mathrm{i}S_{\alpha\beta}(p) = \frac{\mathrm{i}\not{p}}{p^2+\mathrm{i}0}\,\delta_{\alpha\beta}\,, \tag{A.2}$$

  where α, β are the color indices and p is the momentum carried by this line.

- Ghost propagator:

$$\mathrm{i}\widetilde{D}_{ab}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2+\mathrm{i}0}\,, \tag{A.3}$$

  where a, b are the color indices and k is the momentum carried by this line.

- Gluon-quark vertex:

$$\Gamma^{\mu}_{\beta\alpha} = \mathrm{i}g_s\mu^{\epsilon}\gamma^{\mu}t^{a}_{\beta\alpha}\,, \tag{A.4}$$

  where $t^a_{\beta\alpha}$ is the fundamental representation of the generators of the gauge group and a, α and β are the color indices of the gluon, antiquark and quark respectively.

- Three-gluon vertex:

$$G^{\mu_1\mu_2\mu_3}_{a_1a_2a_3}(p_1,p_2,p_3) = \mathrm{i}g_s\mu^{\epsilon}F^{a_1}_{a_2a_3}V^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)\,, \tag{A.5}$$

  where $F^{a_1}_{a_2a_3} = -\mathrm{i}f_{a_1a_2a_3}$ is the adjoint representation of the generators of the gauge group and the kinetic part of the three-gluon vertex is

$$V^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = g^{\mu_3\mu_1}(p_1-p_3)^{\mu_2} + g^{\mu_2\mu_3}(p_3-p_2)^{\mu_1} + g^{\mu_1\mu_2}(p_2-p_1)^{\mu_3}\,, \tag{A.6}$$

  where the p_i momenta are outgoing.

- Four-gluon vertex:

$$\begin{aligned} W^{\mu_1\mu_2\mu_3\mu_4}_{a_1a_2a_3a_4} = \mathrm{i}g_s^2\mu^{2\epsilon}\big(&F^b_{a_1a_2}F^b_{a_3a_4}(g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3}) + \\ &+ F^b_{a_1a_4}F^b_{a_2a_3}(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_3}g^{\mu_2\mu_4}) + \\ &+ F^b_{a_1a_3}F^b_{a_2a_4}(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3})\big)\,, \end{aligned} \tag{A.7}$$

  where the a_i are color indices.

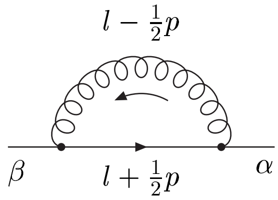

Figure 6: Quark self-energy graph.

- Gluon-ghost vertex:

$$\widetilde{G}^{\mu}_{abc}(k) = \mathrm{i} g_s \mu^{\epsilon} F^{a}_{bc} k^{\mu} \,, \tag{A.8}$$

where a, b, and c are the color indices of the gluon, ghost, and antighost legs respectively and k is the momentum of the outgoing ghost.

We use the standard notation for color factors, $C_A = N_c$ ($= 3$ for SU(3)), $C_F = (N_c^2 - 1)/(2N_c)$, $T_R = 1/2$.

Quark self-energy. The quark self-energy graph is the simplest one-loop Green function. It has logarithmic and linear UV divergences and we can easily define the UV counterterm by applying the prescription introduced in section 4.

This graph is shown in figure 6 and the renormalized quark self-energy is defined by the following integral:

$$\Sigma^{R}_{\alpha\beta}(p) = \int \frac{d^d l}{(2\pi)^d} \left[\widetilde{\Sigma}_{\alpha\beta}(l,p) - \widetilde{\Sigma}^{UV}_{\alpha\beta}(l,p) \right] , \tag{A.9}$$

where the integrand of the self-energy graph is given by

$$\widetilde{\Sigma}_{\alpha\beta}(l,p) = -\mathrm{i} g_s^2 \mu^{2\epsilon} C_F \delta_{\alpha\beta} \frac{\gamma^{\nu} \left(\not{l} + \frac{1}{2}\not{p} \right) \gamma_{\nu}}{\left(\left(l + \frac{1}{2}p\right)^2 + \mathrm{i}0 \right) \left(\left(l - \frac{1}{2}p\right)^2 + \mathrm{i}0 \right)} \,, \tag{A.10}$$

and the corresponding ultra-violet counterterm is

$$\widetilde{\Sigma}^{UV}_{\alpha\beta}(l,p) = -\mathrm{i} g_s^2 \mu^{2\epsilon} C_F \delta_{\alpha\beta} \frac{\gamma^{\nu} \left(\not{l} + \frac{1}{2}\not{p} \right) \gamma_{\nu}}{\left(l^2 - \mu^2 e^{-1} + \mathrm{i}0 \right)^2} \,. \tag{A.11}$$

Here $e = 2.71828\ldots$ is the base of natural logarithms.

Gluon self-energy. The one-loop gluon self-energy is sum of the four contributions shown in figure (7) plus an ultraviolet subtraction,

$$\mathrm{i}\Pi^{\mu\nu\,R}_{ab}(p) = \int \frac{d^d l}{(2\pi)^d} \left[\mathrm{i}\widetilde{\Pi}^{\mu\nu}_{ab}(l,p) - \mathrm{i}\widetilde{\Pi}^{\mu\nu}_{ab\,UV}(l,p) \right] . \tag{A.12}$$

Applying the Feynman rules, one finds that the integrand $\widetilde{\Pi}^{\mu\nu}$ is

$$\begin{aligned}
\widetilde{\Pi}^{\mu\nu}_{ab}(l,p) \;=\; & -\mathrm{i} g_s^2 \mu^{2\epsilon}\, C_A \delta_{ab}\, \frac{1}{2!} \frac{V^{\mu\sigma\delta}(-p,-l_1,l_2) V^{\nu}{}_{\delta\sigma}(p,-l_2,l_1) - l_1^{\mu} l_2^{\nu} - l_2^{\mu} l_1^{\nu}}{(l_1^2 + \mathrm{i}0)(l_2^2 + \mathrm{i}0)} + \\
& + \mathrm{i}\, g_s^2 \mu^{2\epsilon} n_f\, T_R \delta_{ab}\, \frac{\mathrm{Tr}\left[\gamma^{\mu} \not{l}_1 \gamma^{\nu} \not{l}_2 \right]}{(l_1^2 + \mathrm{i}0)(l_2^2 + \mathrm{i}0)} \,,
\end{aligned} \tag{A.13}$$

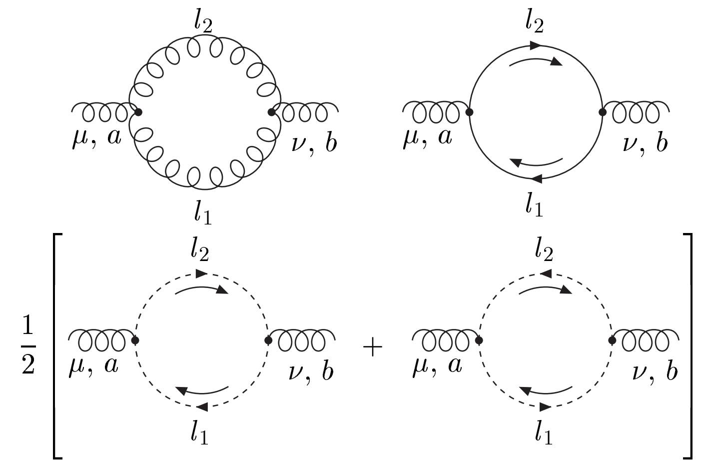

Figure 7: Feynman graphs that contribute to the gluon self-energy.

where a, b are the color indices, $V^{\mu\sigma\delta}$ is the kinematical part of the three-gluon vertex, and the loop integral is parametrized by

$$l_1^\mu = l^\mu - \frac{1}{2}p^\mu\,, \qquad l_2^\mu = l^\mu + \frac{1}{2}p^\mu\,. \tag{A.14}$$

The integrand of the gluon self-energy has quadratic and logarithmic ultraviolet divergences. As pointed out in section 4, the quadratic divergence requires a subtraction defined by a somewhat different prescription than the simple prescription described in general terms in section 4 and used elsewhere in this appendix. We choose the counterterm to be

$$\begin{aligned}\widetilde{\Pi}^{\mu\nu}_{ab\,UV}(l,p) &= -\mathrm{i}g_s^2\mu^{2\epsilon}\delta_{ab}\frac{((d-2)C_A - 4T_Rn_f)\left(2l^\mu l^\nu + \frac{1}{2d-2}(g^{\mu\nu}p^2 - p^\mu p^\nu)\right)}{\left((l-\frac{1}{2}p)^2 + \mathrm{i}0\right)\left((l+\frac{1}{2}p)^2 + \mathrm{i}0\right)} - \\ &\quad -\mathrm{i}g_s^2\mu^{2\epsilon}\delta_{ab}\frac{(C_A + 4T_Rn_f)\left(l^2 + \frac{1}{4}p^2\right)g^{\mu\nu}}{\left((l-\frac{1}{2}p)^2 + \mathrm{i}0\right)\left((l+\frac{1}{2}p)^2 + \mathrm{i}0\right)} - \\ &\quad -\mathrm{i}g_s^2\mu^{2\epsilon}\delta_{ab}\left[\frac{3d-2}{2d-2}\frac{C_A(g^{\mu\nu}p^2 - p^\mu p^\nu)}{(l^2 - \mu^2e^{1/15} + \mathrm{i}0)^2} - \frac{2d-4}{d-1}\frac{T_Rn_f(g^{\mu\nu}p^2 - p^\mu p^\nu)}{(l^2 - \mu^2e^{-1/3} + \mathrm{i}0)^2}\right].\end{aligned} \tag{A.15}$$

The first two terms eliminate all the quadratic divergences and the integral of them is exactly zero in $d = 4 - 2\epsilon$ dimensions. The last term cancels the remaining logarithmic divergences. The integrand $[\widetilde{\Pi}^{\mu\nu}_{ab}(l,p) - \widetilde{\Pi}^{\mu\nu}_{ab\,UV}(l,p)]$ has the familiar tensor structure $g^{\mu\nu}p^2 - p^\mu p^\nu$.

Quark-gluon vertex. At one-loop level there are two graphs that contribute to the quark-gluon vertex, as shown in figure 8. These graphs have only logarithmic ultraviolet divergences. The corresponding renormalized quark-gluon vertex is

$$\begin{aligned}\Gamma^{a\,R}_{\mu\beta\alpha}(p_1,p_2,p_3) &= \mathrm{i}g_s\mu^\epsilon t^a_{\beta\alpha}\gamma_\mu + \mathrm{i}g_s\mu^\epsilon t^a_{\beta\alpha}\int\frac{d^dl}{(2\pi)^d}\left[\widetilde{Q}_\mu(l,p_1,p_2,p_3) - \widetilde{Q}^{UV}_\mu(l)\right] + \\ &\quad +\mathrm{i}g_s\mu^\epsilon t^a_{\beta\alpha}\int\frac{d^dl}{(2\pi)^d}\left[\widetilde{G}_\mu(l,p_1,p_2,p_3) - \widetilde{G}^{UV}_\mu(l)\right].\end{aligned} \tag{A.16}$$

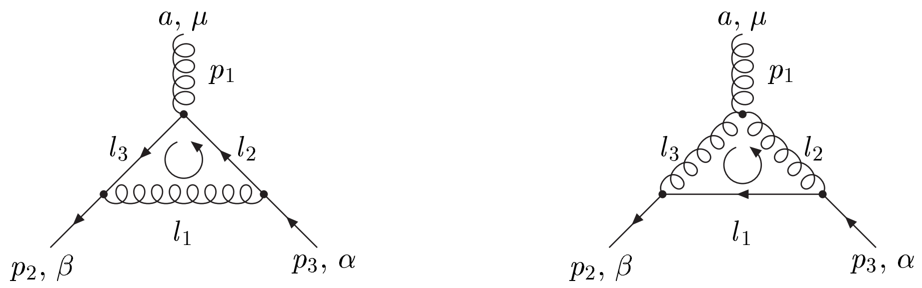

Figure 8: One-loop contributions to the quark-gluon vertex.

The integrands $\widetilde{Q}_\mu$ and $\widetilde{G}_\mu$ are derived directly from the Feynman rules,

$$\begin{aligned}\widetilde{Q}_\mu(l,p_1,p_2,p_3) &= -\mathrm{i}g_s^2\mu^{2\epsilon}\left(C_F-\frac{C_A}{2}\right)\frac{\gamma^\nu \not{l}_3\gamma_\mu\not{l}_2\gamma_\nu}{(l_1^2+\mathrm{i}0)(l_2^2+\mathrm{i}0)(l_3^2+\mathrm{i}0)}\,,\\ \widetilde{G}_\mu(l,p_1,p_2,p_3) &= -\mathrm{i}g_s^2\mu^{2\epsilon}\frac{C_A}{2}\frac{\gamma^\nu \not{l}_1\gamma^\sigma V_{\mu\nu\sigma}(p_1,l_3,-l_2)}{(l_1^2+\mathrm{i}0)(l_2^2+\mathrm{i}0)(l_3^2+\mathrm{i}0)}\,,\end{aligned} \tag{A.17}$$

where $V_{\mu\nu\sigma}$ is the kinematical part of the triple gluon vertex and the loop integral was parametrized by

$$l_1^\mu = l^\mu + \frac{p_3^\mu - p_2^\mu}{3}\,,\qquad l_2^\mu = l^\mu + \frac{p_1^\mu - p_3^\mu}{3}\,,\qquad l_3^\mu = l^\mu + \frac{p_2^\mu - p_1^\mu}{3}\,. \tag{A.18}$$

The corresponding ultraviolet counterterms are

$$\widetilde{Q}_\mu^{UV}(l) = -\mathrm{i}g_s^2\mu^{2\epsilon}(d-2)\left(C_F-\frac{C_A}{2}\right)\frac{l^2\gamma_\mu - 2\not{l}l_\mu}{(l^2-\mu^2e^{-2}+\mathrm{i}0)^3}\,, \tag{A.19}$$

$$\widetilde{G}_\mu^{UV}(l) = -\mathrm{i}g_s^2\mu^{2\epsilon}(d-2)\frac{C_A}{2}\frac{l^2\gamma_\mu(1+\epsilon) + 2\not{l}l_\mu}{(l^2-\mu^2e^{-2/3}+\mathrm{i}0)^3}\,. \tag{A.20}$$

The quark-photon vertex correction can be gotten from the quark-gluon vertex with the substitutions $C_A \to 0$, $C_F \to 1$, $g \to -Q|e|$ and $t^\alpha_{\beta\gamma} \to 1$, where Q is the quark charge in units of $|e|$. Thus the renormalized quark-photon vertex is

$$\Gamma_\mu(p_1,p_2,p_3) = -\mathrm{i}Q|e|\gamma_\mu - \mathrm{i}Q|e|\int\frac{d^dl}{(2\pi)^d}\left[\widetilde{Q}_\mu(l,p_1,p_2,p_3) - \widetilde{Q}_\mu^{UV}(l)\right]_{\substack{C_A=0\\ C_F=1}}\,. \tag{A.21}$$

Three-gluon vertex. The three-gluon vertex is sum of the five triangle graphs and three bubble graphs shown in figure 9. It is useful to group the triangle graphs with a ghost loop together with the triangle graphs with a gluon loop. The other graphs are treated separately. Then the renormalized triple gluon vertex is

$$\begin{aligned}G^{\mu_1\mu_2\mu_3\,R}_{a_1a_2a_3}(p_1,p_2,p_3) &= \mathrm{i}g_s\mu^{2\epsilon}F^{a_1}_{a_2a_3}V^{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) + \mathrm{i}g_s\mu^{2\epsilon}F^{a_1}_{a_2a_3}\int\frac{d^dl}{(2\pi)^d}\times\\ &\times\left[\widetilde{T}^{\mu_1\mu_2\mu_3}(l,p_1,p_2,p_3) - \widetilde{T}^{\mu_1\mu_2\mu_3}_{UV}(l,p_1,p_2,p_3)\right] +\end{aligned} \tag{A.22}$$

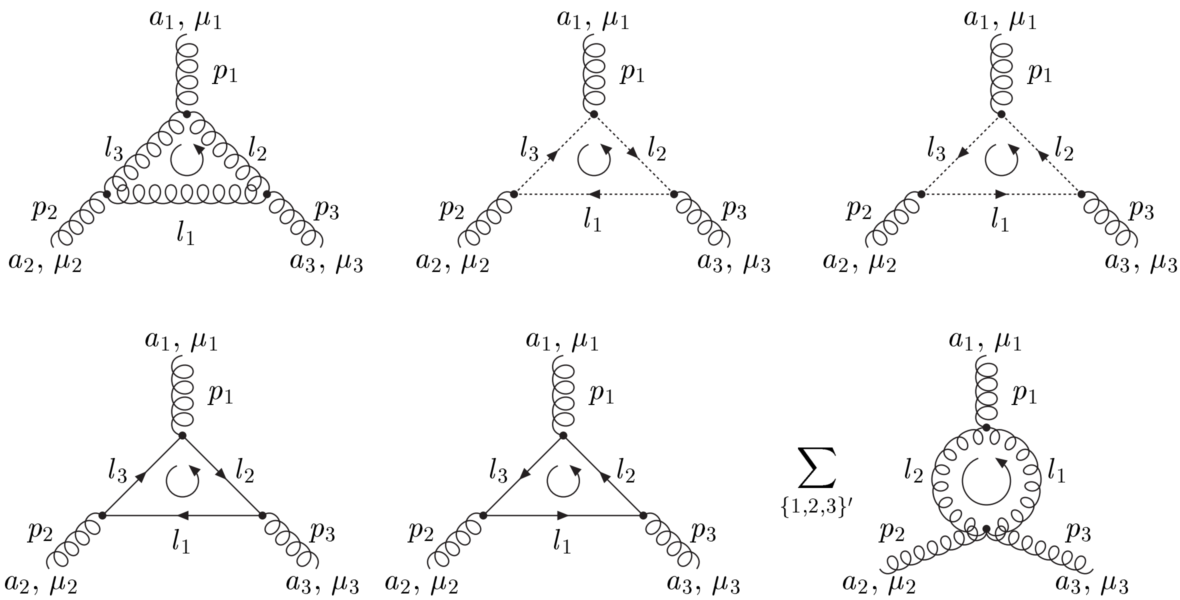

Figure 9: One-loop corrections to the three-gluon vertex. The $\{1,2,3\}'$ denotes the cyclic permutation of the indices $(1,2,3)$.

$$+\sum_{\{1,2,3\}'} \mathrm{i}g_s\mu^{2\epsilon}F^{a_1}_{a_2a_3}\int\frac{d^dl}{(2\pi)^d}\left[\widetilde{B}^{\mu_1\mu_2\mu_3}(l,p_1)-\widetilde{B}^{\mu_1\mu_2\mu_3}_{UV}(l,p_1)\right],$$

where $\widetilde{T}^{\mu_1\mu_2\mu_3}$ is the contribution of all the triangle graphs, $\widetilde{B}^{\mu_1\mu_2\mu_3}$ represents one of the bubble graphs, and $\{1,2,3\}'$ denotes the cyclic permutations of the indices $(1,2,3)$. The integrand of the triangle graphs is given by

$$\begin{aligned}\widetilde{T}^{\mu_1\mu_2\mu_3}(l,p_1,p_2,p_3) = -\mathrm{i}g_s^2\mu^{2\epsilon}\Bigg\{&\frac{C_A}{2}\,\frac{V^{\mu_1\delta}{}_{\lambda}(p_1,l_3,-l_2)V^{\mu_2\gamma}{}_{\delta}(p_2,l_1,-l_3)V^{\mu_3\lambda}{}_{\gamma}(p_3,l_2,-l_1)}{(l_1^2+\mathrm{i}0)\,(l_2^2+\mathrm{i}0)\,(l_3^2+\mathrm{i}0)}+\\ &+\frac{C_A}{2}\,\frac{l_3^{\mu_1}l_1^{\mu_2}l_2^{\mu_3}+l_2^{\mu_1}l_3^{\mu_2}l_1^{\mu_3}}{(l_1^2+\mathrm{i}0)\,(l_2^2+\mathrm{i}0)\,(l_3^2+\mathrm{i}0)}+\\ &+T_Rn_f\,\frac{\mathrm{Tr}\left[\gamma^{\mu_1}\not{l}_3\gamma^{\mu_2}\not{l}_1\gamma^{\mu_3}\not{l}_2\right]}{(l_1^2+\mathrm{i}0)\,(l_2^2+\mathrm{i}0)\,(l_3^2+\mathrm{i}0)}\Bigg\}.\end{aligned} \tag{A.23}$$

The loop integral here is parametrized as

$$l_1^\mu = l^\mu+\frac{p_3^\mu-p_2^\mu}{3},\qquad l_2^\mu = l^\mu+\frac{p_1^\mu-p_3^\mu}{3},\qquad l_3^\mu = l^\mu+\frac{p_2^\mu-p_1^\mu}{3}. \tag{A.24}$$

With these choices, the ultraviolet counterterm is given by

$$\widetilde{T}^{\mu_1\mu_2\mu_3}_{UV}(l) = -\mathrm{i}g_s^2\mu^{2\epsilon}\left(\frac{C_A}{2}\frac{N_g^{\mu_1\mu_2\mu_3}(l)}{(l^2-\mu^2e^{-11/19}+\mathrm{i}0)^3}+T_R\,n_f\,\frac{N_q^{\mu_1\mu_2\mu_3}(l)}{(l^2-\mu^2e^{-1}+\mathrm{i}0)^3}\right), \tag{A.25}$$

where the numerator functions are

$$N_g^{\mu_1\mu_2\mu_3}(l) = \sum_{\{1,2,3\}'}\Bigg(-2l^2g^{\mu_1\mu_2}l^{\mu_3}-\frac{8}{3}(d-2)l^{\mu_1}l^{\mu_2}l^{\mu_3}+\frac{4}{3}(d-2)l^{\mu_1}l^{\mu_2}(p_2-p_1)^{\mu_3}+$$

$$+\frac{1}{3}g^{\mu_1\mu_2}\left(7\,l^2(p_2-p_1)^{\mu_3}+2\,l\cdot(p_2-p_1)l^{\mu_3}\right)\Big) \tag{A.26}$$

and

$$N_q^{\mu_1\mu_2\mu_3}(l) = \sum_{\{1,2,3\}'}\Big(-4l^2g^{\mu_1\mu_2}l^{\mu_3}+\frac{16}{3}l^{\mu_1}l^{\mu_2}l^{\mu_3}-\frac{8}{3}l^{\mu_1}l^{\mu_2}(p_2-p_1)^{\mu_3}+ \\ +\frac{8}{3}g^{\mu_1\mu_2}\left(l^2(p_2-p_1)^{\mu_3}-l\cdot(p_2-p_1)l^{\mu_3}\right)\Big)\,. \tag{A.27}$$

The bubble integral is simpler. In this case the integrand has only a logarithmic divergence. The integrand is given by

$$\widetilde{B}^{\mu_1\mu_2\mu_3}(l_1,l_2,p_1) = \mathrm{i}g_s^2\mu^{2\epsilon}\frac{9}{4}C_A\frac{g^{\mu_1\mu_3}p_1^{\mu_2}-g^{\mu_1\mu_2}p_1^{\mu_3}}{(l_1^2+\mathrm{i}0)(l_2^2+\mathrm{i}0)}\,, \tag{A.28}$$

with the following parametrization for the loop integral,

$$l_1^\mu = l^\mu+\frac{1}{2}p_1^\mu\,,\qquad l_2^\mu = l^\mu-\frac{1}{2}p_1^\mu\,. \tag{A.29}$$

The corresponding counterterm for the bubble integrals is

$$\widetilde{B}_{UV}^{\mu_1\mu_2\mu_3}(l,p_1) = \mathrm{i}g_s^2\mu^{2\epsilon}\frac{9}{4}C_A\frac{g^{\mu_1\mu_3}p_1^{\mu_2}-g^{\mu_1\mu_2}p_1^{\mu_3}}{(l^2-\mu^2+\mathrm{i}0)^2}\,. \tag{A.30}$$

Four-gluon vertex. The one-loop correction to the four-gluon vertex is sum of box, triangle, and bubble graphs. In the case of the box graphs, it is useful to group the graphs with a ghost loop together with the graphs with a gluon loop. We write the renormalized vertex as

$$Q^{\mu_1\mu_2\mu_3\mu_4}_{a_1a_2a_3a_4\,R}(p_1,p_2,p_3,p_4) = \mathrm{i}g_s^2\mu^{2\epsilon}W^{\mu_1\mu_2\mu_3\mu_4}_{a_1a_2a_3a_4}+\mathrm{i}g_s^2\mu^{2\epsilon}\sum_{P_1}\int\frac{d^dl}{(2\pi)^d}\times \\ \times\left[\widetilde{B}^{\mu_1\mu_2\mu_3\mu_4}_{a_1a_2a_3a_4}(l,p_1,p_2,p_3,p_4)-\widetilde{B}^{\mu_1\mu_2\mu_3\mu_4}_{UV\,a_1a_2a_3a_4}(l)\right]+ \\ +\mathrm{i}g_s^2\mu^{2\epsilon}\sum_{P_2}\int\frac{d^dl}{(2\pi)^d}\times \\ \times\left[\widetilde{T}^{\mu_1\mu_2\mu_3\mu_4}_{a_1a_2a_3a_4}(l,p_1,p_2,p_3,p_4)-\widetilde{T}^{\mu_1\mu_2\mu_3\mu_4}_{UV\,a_1a_2a_3a_4}(l)\right]+ \\ +\mathrm{i}g_s^2\mu^{2\epsilon}\sum_{P_3}\int\frac{d^dl}{(2\pi)^d}\times \\ \times\left[\widetilde{F}^{\mu_1\mu_2\mu_3\mu_4}_{a_1a_2a_3a_4}(l,p_1,p_2,p_3,p_4)-\widetilde{F}^{\mu_1\mu_2\mu_3\mu_4}_{UV\,a_1a_2a_3a_4}(l)\right]\,, \tag{A.31}$$

where the P_1, P_2 and P_3 are certain sets of permutations of the indices $(1,2,3,4)$,

$$\begin{aligned} P_1 &= \{(1,2,3,4),(1,2,4,3),(1,3,2,4)\}\,, \\ P_2 &= \{(1,2,3,4),(1,4,3,2),(1,3,2,4),(4,2,3,1),(3,2,1,4),(4,3,2,1)\}\,, \\ P_3 &= \{(1,2,3,4),(1,3,2,4),(1,4,3,2)\}\,. \end{aligned} \tag{A.32}$$

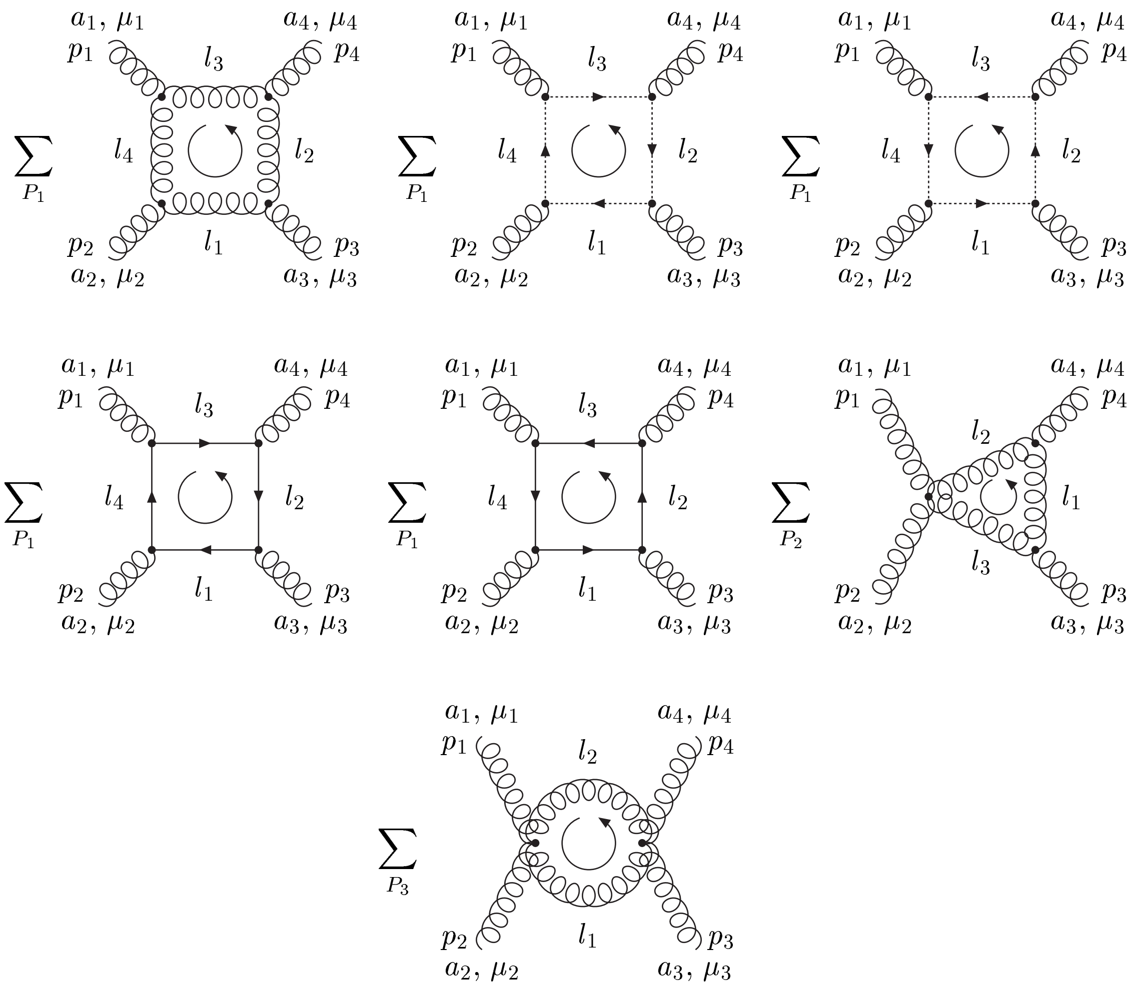

Figure 10: One-loop level correction to the four-point gluon vertex. The P_1, P_2, P_3 denote the permutations of the indices $(1, 2, 3, 4)$ and they are given in eq. (A.32).

The integrand of the box diagrams can be obtained by applying the Feynman rules,

$$\widetilde{B}^{\mu_1\mu_2\mu_3\mu_4}_{a_1a_2a_3a_4}(l, p_1, p_2, p_3, p_4) = -\mathrm{i}g_s^2\mu^{2\epsilon}\frac{\mathrm{Tr}\,(F^{a_1}F^{a_2}F^{a_3}F^{a_4})\,N_g^{\mu_1\mu_2\mu_3\mu_4}(l_1, l_2, l_3, l_4)}{\left(l_1^2+\mathrm{i}0\right)\left(l_2^2+\mathrm{i}0\right)\left(l_3^2+\mathrm{i}0\right)\left(l_4^2+\mathrm{i}0\right)}+$$
$$+\mathrm{i}g_s^2\mu^{2\epsilon}\frac{2n_f\,\mathrm{Tr}\,(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\,N_q^{\mu_1\mu_2\mu_3\mu_4}(l_1, l_2, l_3, l_4)}{\left(l_1^2+\mathrm{i}0\right)\left(l_2^2+\mathrm{i}0\right)\left(l_3^2+\mathrm{i}0\right)\left(l_4^2+\mathrm{i}0\right)}\,, \quad \text{(A.33)}$$

where the loop integral is parametrized by

$$l_1^\mu = l^\mu + \frac{2p_3^\mu + p_4^\mu - p_2^\mu}{4}\,, \qquad l_2^\mu = l^\mu - \frac{2p_3^\mu + p_2^\mu - p_4^\mu}{4}\,,$$
$$l_3^\mu = l^\mu + \frac{2p_1^\mu + p_2^\mu - p_4^\mu}{4}\,, \qquad l_4^\mu = l^\mu - \frac{2p_1^\mu + p_4^\mu - p_2^\mu}{4}\,, \qquad \text{(A.34)}$$

and the numerator functions N_g and N_q are given by

$$\begin{aligned} N_g^{\mu_1\mu_2\mu_3\mu_4}(l_1,l_2,l_3,l_4) &= V^{\mu_1\alpha}{}_{\delta}(l_3-l_4,l_4,-l_3)V^{\mu_2\beta}{}_{\alpha}(l_4-l_1,l_1,-l_4)\times \\ &\qquad \times V^{\mu_3\gamma}{}_{\beta}(l_1-l_2,l_2,-l_1)V^{\mu_4\delta}{}_{\gamma}(l_2-l_3,l_3,-l_2)- \\ &\qquad -l_4^{\mu_1}l_1^{\mu_2}l_2^{\mu_3}l_3^{\mu_4}-l_3^{\mu_1}l_4^{\mu_2}l_1^{\mu_3}l_2^{\mu_4}\,, \qquad (A.35) \\ N_q^{\mu_1\mu_2\mu_3\mu_4}(l_1,l_2,l_3,l_4) &= \mathrm{Tr}\left[\gamma^{\mu_1}\not{l}_4\gamma^{\mu_2}\not{l}_1\gamma^{\mu_3}\not{l}_2\gamma^{\mu_4}\not{l}_3\right]. \qquad (A.36) \end{aligned}$$

The integral of the box graphs has only a logarithmic divergence. The ultraviolet counterterm is

$$\begin{aligned} \widetilde{B}^{\mu_1\mu_2\mu_3\mu_4}_{UV\,a_1a_2a_3a_4}(l) &= -\mathrm{i}g_s^2\mu^{2\epsilon}\frac{\mathrm{Tr}\left(F^{a_1}F^{a_2}F^{a_3}F^{a_4}\right)N_g^{\mu_1\mu_2\mu_3\mu_4}(l,l,l,l)}{\left(l^2-\mu^2e^{-2/3}+\mathrm{i}0\right)^4}+ \\ &\quad +\mathrm{i}g_s^2\mu^{2\epsilon}\frac{2n_f\,\mathrm{Tr}\left(t^{a_1}t^{a_2}t^{a_3}t^{a_4}\right)\left(N_q^{\mu_1\mu_2\mu_3\mu_4}(l,l,l,l)-32\,l^{\mu_1}l^{\mu_2}l^{\mu_3}l^{\mu_4}\right)}{\left(l^2-\mu^2e^{-4/3}+\mathrm{i}0\right)^4}+ \\ &\quad +\mathrm{i}g_s^2\mu^{2\epsilon}\frac{64n_f\,\mathrm{Tr}\left(t^{a_1}t^{a_2}t^{a_3}t^{a_4}\right)l^{\mu_1}l^{\mu_2}l^{\mu_3}l^{\mu_4}}{\left(l^2-\mu^2e^{-3/2}+\mathrm{i}0\right)^4}\,. \qquad (A.37) \end{aligned}$$

The integrand of triangle graphs is

$$\widetilde{T}^{\mu_1\mu_2\mu_3\mu_4}_{a_1a_2a_3a_4}(l,p_1,p_2,p_3,p_4)=-\mathrm{i}g_s^2\mu^{2\epsilon}\frac{F^{a_3}_{ca}F^{a_4}_{bc}W^{\mu_1\mu_2\alpha\beta}_{a_1a_2a\,b}V^{\mu_4}{}_{\beta\gamma}(p_4,l_2,-l_1)V^{\mu_3\gamma}{}_{\alpha}(p_3,l_1,-l_3)}{\left(l_1^2+\mathrm{i}0\right)\left(l_2^2+\mathrm{i}0\right)\left(l_3^2+\mathrm{i}0\right)}\,, \qquad (A.38)$$

where the loop integral is parametrized by

$$l_1^\mu=l^\mu+\frac{p_4^\mu-p_3^\mu}{3}\,,\qquad l_2^\mu=l^\mu+\frac{p_1^\mu+p_2^\mu-p_4^\mu}{3}\,,\qquad l_3^\mu=l^\mu+\frac{p_3^\mu-p_1^\mu-p_2^\mu}{3}\,. \qquad (A.39)$$

With this parametrization the UV counterterm to the triangle graph is

$$\widetilde{T}^{\mu_1\mu_2\mu_3\mu_4}_{UV\,a_1a_2a_3a_4}(l)=-\mathrm{i}g_s^2\mu^{2\epsilon}\frac{F^{a_3}_{ca}F^{a_4}_{bc}W^{\mu_1\mu_2\alpha\beta}_{a_1a_2a\,b}V^{\mu_4}{}_{\beta\gamma}(0,l,-l)V^{\mu_3\gamma}{}_{\alpha}(0,l,-l)}{\left(l^2-\mu^2e^{-2/9}+\mathrm{i}0\right)^3}\,. \qquad (A.40)$$

Applying the Feynman rules, the integrand of the bubble graphs can be found easily,

$$\widetilde{F}^{\mu_1\mu_2\mu_3\mu_4}_{a_1a_2a_3a_4}(l,p_1,p_2,p_3,p_4)=-\mathrm{i}g_s^2\mu^{2\epsilon}\frac{1}{2!}\frac{g_{\alpha\alpha'}g_{\beta\beta'}W^{\mu_1\mu_2\alpha\beta}_{a_1a_2a\,b}W^{\beta'\alpha'\mu_3\mu_4}_{b\,a\,a_3a_4}}{\left(l_1^2+\mathrm{i}0\right)\left(l_2^2+\mathrm{i}0\right)}\,, \qquad (A.41)$$

where

$$l_1^\mu=l^\mu-\frac{p_1^\mu+p_2^\mu}{2}\,,\qquad l_2^\mu=l^\mu+\frac{p_1^\mu+p_2^\mu}{2}\,. \qquad (A.42)$$

With this, the UV counterterm to a bubble graph is

$$\widetilde{F}^{\mu_1\mu_2\mu_3\mu_4}_{UV\,a_1a_2a_3a_4}(l)=-\mathrm{i}g_s^2\mu^{2\epsilon}\frac{1}{2!}\frac{g_{\alpha\alpha'}g_{\beta\beta'}W^{\mu_1\mu_2\alpha\beta}_{a_1a_2a\,b}W^{\beta'\alpha'\mu_3\mu_4}_{b\,a\,a_3a_4}}{\left(l^2-\mu^2e^{1/3}+\mathrm{i}0\right)^2}\,. \qquad (A.43)$$

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